AXIOMS FOR VECTOR SPACES
MATH 108A, Summer 2008

1 Field axioms

Definition. A *field* is a set F together with two operations (functions)

$$f : F \times F \to F, \qquad f(x, y) = x + y$$

and

$$g : F \times F \to F, \qquad g(x, y) = xy,$$

which satisfy the following axioms:

1. addition is commutative: $x + y = y + x$, for all $x, y \in F$.
2. addition is associative: $(x + y) + z = x + (y + z)$, for all $x, y, z \in F$.
3. existence of additive identity: there is an element $0 \in F$ such that $x + 0 = x$, for all $x \in F$.
4. existence of additive inverses: if $x \in F$, there is an element $-x \in F$ such that $x + (-x) = 0$.
5. multiplication is commutative: $xy = yx$, for all $x, y \in F$.
6. multiplication is associative: $(xy)z = x(yz)$, for all $x, y, z \in F$.
7. existence of multliplicative identity: there is an element $1 \in F$ such that $1 \neq 0$ and $x1 = x$, for all $x \in F$.
8. existence of multliplicative inverses: if $x \in F$ and $x \neq 0$, there is an element $(1/x) \in F$ such that $x(1/x) = 1$.
9. distributivity: $x(y + z) = xy + xz$, for all $x, y, z \in F$.

Examples: The rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$ are all fields, when f and g are the usual addition and multiplication operations. Note, however, that the integers $\mathbb{Z}$ with the usual addition and multiplication operations is NOT a field because the quotient of two integers is not always an integer.

2 Vector space axioms

Definition. Suppose that F is a field. A *vector space* over F is a set V together with two operations (functions)

$$f : V \times V \to V, \qquad f(\mathbf{v}, \mathbf{w}) = \mathbf{v} + \mathbf{w}$$

and

$$g : F \times V \to V, \qquad g(a, \mathbf{v}) = a\mathbf{v},$$

called vector addition and scalar multiplication, which satisfy the following axioms:

1. vector addition is commutative: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$, for all $\mathbf{u}, \mathbf{v} \in V$.
2. vector addition is associative: $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$, for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$.
3. existence of additive identity: there is an element $\mathbf{0} \in V$ such that $\mathbf{v}+\mathbf{0} = \mathbf{v}$, for all $\mathbf{v} \in V$.
4. existence of additive inverses: if $\mathbf{v} \in V$, there is an element $\mathbf{w} \in V$ such that $\mathbf{v} + \mathbf{w} = \mathbf{0}$.
5. scalar multiplication is associative: $(ab)\mathbf{v} = a(b\mathbf{v})$, for all $a, b \in F$, $\mathbf{v} \in V$.
6. multliplicative identity: $1\mathbf{v} = \mathbf{v}$, for all $\mathbf{v} \in V$.
7. distributivity 1: $a(u + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$, for all $a \in F$, $\mathbf{u}, \mathbf{v} \in V$.
8. distributivity 2: $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$ for all $a, b \in F$, $\mathbf{v} \in V$.

A vector space over the field $\mathbb{Q}$ is called a *rational vector space*. A vector space over $\mathbb{R}$ is called a *real vector space*. A vector space over $\mathbb{C}$ is called a *complex vector space*.

Examples: If F is a field and n is a positive integer, we let F^n denote the set of lists of elements of F of length n. If

$$\mathbf{x} = (x_1, \ldots, x_n) \qquad \text{and} \qquad \mathbf{y} = (y_1, \ldots, y_n)$$

are elements of F^n, we define vector addition by

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \ldots, x_n + y_n).$$

If $a \in F$, we define scalar multiplication by

$$a\mathbf{x} = (ax_1, \ldots, ax_n).$$

We can also let F^∞ be the set of infinite sequences of elements of F. If

$$\mathbf{x} = (x_1, x_2, \ldots) \qquad \text{and} \qquad \mathbf{y} = (y_1, y_2, \ldots)$$

are elements of F^∞, we define vector addition by

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \ldots).$$

If $a \in F$, we define scalar multiplication by

$$a\mathbf{x} = (ax_1, ax_2, \ldots).$$

3 Inner product space axioms

Definition. Suppose that $F = \mathbb{R}$ or $F = \mathbb{C}$. An *inner product space* over F is a vector space over F together with an operation (function)

$$f : V \times V \to F, \qquad f(\mathbf{v}, \mathbf{w}) = \langle \mathbf{v}, \mathbf{w} \rangle$$

which satisfies the following axioms:

1. positivity: $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$, for all $\mathbf{v} \in V$, and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$, if and only if $\mathbf{v} = \mathbf{0}$.
2. linearity in first variable: $\langle a\mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = a\langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$, for all $a \in F$, $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$.
3. symmetry: $\langle \mathbf{v}, \mathbf{u} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$, for all $\mathbf{u}, \mathbf{v} \in V$ if $F = \mathbb{R}$, or $\langle \mathbf{v}, \mathbf{u} \rangle = \overline{\langle \mathbf{u}, \mathbf{v} \rangle}$, for all $\mathbf{u}, \mathbf{v} \in V$ if $F = \mathbb{C}$.

In the latter case, $\overline{\langle \mathbf{u}, \mathbf{v} \rangle}$ denotes the complex conjugate of the complex number $\langle \mathbf{u}, \mathbf{v} \rangle$.

Examples: If $F = \mathbb{R}$ and

$$\mathbf{x} = (x_1, \ldots, x_n), \qquad \mathbf{y} = (y_1, \ldots, y_n)$$

are elements of $\mathbb{R}^n$, we let

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = (x_1 y_1, \ldots, x_n y_n).$$

This inner product is known as the *dot product* on $\mathbb{R}^n$. If $F = \mathbb{C}$ and

$$\mathbf{x} = (x_1, \ldots, x_n), \qquad \mathbf{y} = (y_1, \ldots, y_n)$$

are elements of $\mathbb{C}^n$, we let

$$\langle \mathbf{x}, \mathbf{y} \rangle = (x_1 \bar{y}_1, \ldots, x_n \bar{y}_n).$$