1. Complex Numbers

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These notes are intended to supplement the text, *Fundamentals of complex analysis*, by Saff and Snider [5]. Other often-used references for the theory of analytic functions of a complex variable are the alternate text by Churchill and Brown [2], and the more advanced classic by Ahlfors [1]. For a history of the development of complex numbers, we recommend relevant chapters of [3].

If we were to develop real and complex analysis from the foundation up, we would start with set theory (as studied in Math 8). Using sets, we would build up successively the natural numbers, the integers, the rational numbers and the real numbers.

We would start by defining the set of *natural numbers*

$$\mathbb{N} = \{1, 2, 3, \ldots\} \quad \text{and} \quad \omega = \{0\} \cup \mathbb{N} = \{0, 1, 2, 3, \ldots\},$$

together with the usual operations of addition and multiplication in terms of sets. We could then define an equivalence relation $\sim$ on the Cartesian product $\omega \times \omega$ by

$$(m, n) \sim (q, r) \quad \Leftrightarrow \quad m + r = q + n.$$

The equivalence class $[m, n]$ of the pair $(m, n) \in \omega \times \omega$ would be thought of as the difference $m - n$, and the set of equivalence classes

$$\mathbb{Z} = \{\ldots, -2 = [0, 2], -1 = [0, 1], 0 = [0, 0], 1 = [1, 0], 2 = [2, 0], \ldots\}$$

would then be regarded as the set of *integers*. We would next define the usual addition and multiplication on $\mathbb{Z}$ and show that these operations satisfy the familiar properties. The advantage of $\mathbb{Z}$ over $\mathbb{N}$ is that subtraction is always defined.

Next, we would define an equivalence relation $\sim$ on the Cartesian product $\mathbb{Z} \times (\mathbb{Z} - \{0\})$ by

$$(m, n) \sim (q, r) \quad \Leftrightarrow \quad mr = qn,$$

and think of the equivalence class $[m, n]$ are representing the fraction m/n. The set of all such fractions is known as the set $\mathbb{Q}$ of rational numbers. We could then define addition and multiplication by

$$\frac{m}{n} + \frac{q}{r} = \frac{mr + qn}{nr}, \quad \frac{m}{n} \cdot \frac{q}{r} = \frac{mq}{nr},$$

and establish all the usual rules of arithmetic with rational numbers, familiar from grade school, including now division. A complete construction would be long and time-consuming, and you might wonder whether it isn't a bit pedantic to carry this out with so much rigor. But it is important to understand that all of the familiar rules of arithmetic for rational numbers can in fact be established by deduction from the axioms of set theory.

The last stage is developing the real numbers $\mathbb{R}$, which can be thought of as limits of sequences of rational numbers. For example, the number π is the limit of the sequence

$$(3, 3.1, 3.14, 3.141, 3.1415, 3.14159, 3.141592, \ldots., 3.14159265358979, \ldots).$$

It is precisely the notion of defining the limit of such a sequence which is the major difficulty in developing real analysis. It would take a long time just to define the real numbers in this manner, so for a first treatment of real analysis, most authors take a shortcut, and formulate a collection of axioms which characterize the real numbers. One often assumes these axioms as the starting point of real analysis, rather than just the axioms of set theory. (Since one does want to use the properties of sets in discussing real numbers, a full formal development of analysis in this shortened form would require both the axioms of set theory and the axioms of real numbers. On the other hand, many authors, just use set theory as a basic language whose basic properties are intuitively clear; this is more or less the way mathematicians thought about set theory prior to its study by Georg Cantor (1845-1918) and its later axiomatization.)

The rational numbers and the real numbers both satisfy the field axioms that we next describe.

1 Field axioms

Definition. A *field* is a set F together with two operations (functions)

$$f : F \times F \to F, \qquad f(x, y) = x + y$$

and

$$g : F \times F \to F, \qquad g(x, y) = xy,$$

called addition and multiplication, respectively, which satisfy the following field axioms:

- F1. addition is commutative: $x + y = y + x$, for all $x, y \in F$.
- F2. addition is associative: $(x + y) + z = x + (y + z)$, for all $x, y, z \in F$.
- F3. existence of additive identity: there is a unique element $0 \in F$ such that $x + 0 = x$, for all $x \in F$.
- F4. existence of additive inverses: if $x \in F$, there is a unique element $-x \in F$ such that $x + (-x) = 0$.

- F5. multiplication is commutative: $xy = yx$, for all $x, y \in F$.
- F6. multiplication is associative: $(xy)z = x(yz)$, for all $x, y, z \in F$.
- F7. existence of multliplicative identity: there is a unique element $1 \in F$ such that $1 \neq 0$ and $x1 = x$, for all $x \in F$.
- F8. existence of multliplicative inverses: if $x \in F$ and $x \neq 0$, there is a unique element $(1/x) \in F$ such that $x \cdot (1/x) = 1$.
- F9. distributivity: $x(y + z) = xy + xz$, for all $x, y, z \in F$.

Note the similarity between axioms F1-F4 and axioms F5-F8. In the language of algebra, axioms F1-F4 state that F with the addition operation f is an *abelian group*. (The group axioms are studied further in the first part of abstract algebra, which is devoted to group theory.) Axioms F5-F8 state that $F - \{0\}$ with the multiplication operation g is also an abelian group. Axiom F9 ties the two field operations together.

Among the most important examples of fields are the set of rational numbers $\mathbb{Q}$ and the set of real numbers $\mathbb{R}$. In both cases we take f and g to be the usual addition and multiplication operations. On the other hand, the set of integers $\mathbb{Z}$ is NOT a field, because integers do not always have multiplicative inverses.

The field of reals $\mathbb{R}$ is much larger than the field $\mathbb{Q}$ of rationals. Indeed, as you have most likely seen in Math 8, Georg Cantor proved that the field $\mathbb{Q}$ of rational numbers is countable, that is, in one-to-one correspondence with $\mathbb{N}$, while the field $\mathbb{R}$ is uncountable.

Another example. We can define a field $\mathbb{Z}/p\mathbb{Z}$, where p is a prime ≥ 2, which consists of the elements $\{0, 1, 2, \ldots, p-1\}$. In this case, we define addition or multiplication by first forming the sum or product in the usual sense and then taking the remainder after division by p, so as to arrive back in the set $\{0, 1, 2, \ldots, p-1\}$. This is often referred to as mod p addition and multiplication. Thus for example,

$$\mathbb{Z}/5\mathbb{Z} = \{0, 1, 2, 3, 4\}$$

and within $\mathbb{Z}/5\mathbb{Z}$,

$$3 + 4 = 7 \bmod 5 = 2, \qquad 3 \cdot 4 = 12 \bmod 5 = 2.$$

One can use theorems on prime factorization to show that $\mathbb{Z}/p\mathbb{Z}$ satisfies all the field axioms. On the other hand, if n is not a prime, then $\mathbb{Z}/n\mathbb{Z}$ with mod n addition and multiplication is NOT a field. Indeed, in $\mathbb{Z}/4\mathbb{Z}$,

$$2 + 2 = 4 \bmod 4 = 0,$$

so 2 does not have a multiplicative inverse in $\mathbb{Z}/4\mathbb{Z}$, contradicting Axiom F8.

2 Complex numbers

Unfortunately, it is not possible to take the square roots of a negative real number and get a real number as a result. This defect makes it impossible to find solutions to polynomial equations like

$$x^2 + 1 = 0$$

when using just real numbers. In order to remedy this problem, we introduce the complex numbers $\mathbb{C}$. There are two common ways of doing this:

Method I. We can utilize the theory of matrices, and regard the space $\mathbb{C}$ of complex numbers to be the set of 2×2 matrices of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

where a and b are real numbers. One can check that the sum or product of two elements of $\mathbb{C}$ is again an element of $\mathbb{C}$. Although matrix do not commute in general, it is the case that

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} c & -d \\ d & c \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

for any choice of a, b, c and d, as you can easily verify by direct multiplication. We often use the notation

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

so that

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a + bi.$$

The set $\mathbb{C}$ of matrices we have described, together with the operations of matrix addition and multiplication, satisfies the field axioms, and we can call it the field of complex numbers.

Method II. We can also think of the space $\mathbb{C}$ of complex numbers as the space $\mathbb{R}^2$ of ordered pairs of real numbers (a, b) using vector addition for addition, with the additional structure of a multiplication defined by the formula

$$(a, b)(c, d) = (ac - bd, ad + bc). \tag{1}$$

Alternatively, we can set

$$1 = (1, 0), \quad i = (0, 1), \quad \text{so} \quad (a, b) = a + bi,$$

and the formula (1) then shows that $i^2 = -1$. Once again, the set $\mathbb{C}$ of complex numbers is a field under the addition and multiplication operations that we have defined. Thus all of the usual rules of arithmetic (such as the associative, commutative and distributive laws) can be applied to complex numbers.

The second approach is the one adopted by the text [5], and suggests an important way of visualizing complex numbers. A complex number

$$z = x + iy$$

can be thought of as representing a point in the (x, y)-plane. We say that x is the *real part* of z, while y is the *imaginary part*, and we write

$$x = \mathrm{Re}(z), \quad y = \mathrm{Im}(z).$$

Using the Pythagorean law, we can define the length or *modulus* of the complex number $z = x + iy$ by

$$|z| = \sqrt{x^2 + y^2}.$$

Of the main operations on complex numbers, only division might provide a challenge for calculation when starting out. If $a + ib$ is a complex number, its *conjugate* is

$$\overline{a + ib} = a - ib.$$

The division of complex numbers is then obtained by multiplying both numerator and denominator by the conjugate of the denominator:

$$\frac{3 + 5i}{2 + 3i} = \frac{3 + 5i}{2 + 3i}\frac{2 - 3i}{2 - 3i} = \frac{21 + i}{13}.$$

The complex numbers provide an important extension of the real numbers, because within the complex numbers, one can always solve quadratic equations. Recall that if $a, b, c \in \mathbb{R}$, the roots of the quadratic equations

$$az^2 + bz + c = 0 \qquad \text{are} \qquad z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{2}$$

The solutions can always be written as complex numbers, because we can always find a square root of $b^2 - 4ac$, even if it is negative.

It is with the quadratic formula (2) that students often encounter complex numbers for the first time. Although they at first appear strange, it soon becomes apparent that the complex numbers often pay for themselves many times over in finding a solution within the real numbers to a problem expressed in terms of the real numbers.

Of course, one might try to find a similar formula for zeros of the cubic

$$az^3 + bz^2 + cz + d = 0.$$

Finding such a formula was one of the successes of the Renaissance mathematicians in Italy, a solution in terms of radicals being found by Tartaglia and Cardano in 1545. Although these Italian mathematicians did not use complex numbers, their cumbersome methods are far easier to understand with complex numbers. An even more complicated formula was found for zeros of quartics. Évariste Galois (1811-32) was able to show that there is no formula in terms of radicals for zeros to the most general quintic polynomials.

Nevertheless, a far-reaching existence theorem on roots to polynomials is often attributed to Carl Friedrich Gauss (1777-1855):

Fundamental Theorem of Algebra. *Every nonconstant polynomial with complex coefficients has at least one complex zero.*

When the degree of the polynomial is large, one is usually forced to use numerical methods to find approximations to the zeros. The Fundamental Theorem provides one of the main reasons for the importance of complex analysis. Hopefully, we will give a proof of this important result later in the course.

Quaternions. One might wonder whether it is possible to extend the notion of complex numbers yet again to a larger field. This was tried by Sir William Rowen Hamilton (1805-65) who developed the quaternions as a result; see [3], 776-782. In modern notation, we would define the space $\mathbb{H}$ of *quaternions* to be the set of 2×2 matrices of the form

$$\begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix},$$

where z and w are complex numbers with conjugates $\bar{z}$ and $\bar{w}$. Once again, one can check that the sum or product of two elements of $\mathbb{H}$ is again in $\mathbb{H}$. The operations of matrix addition and multiplication satisfy all of the field operations except for commutativity of multiplication F5. Indeed, one can check that if

$$\mathbf{i} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

then

$$\mathbf{ij} = \mathbf{k}, \quad \text{while} \quad \mathbf{ji} = -\mathbf{k},$$

in analogy with the cross product. Thus the sets of quaternions $\mathbb{H}$ is not quite a field, but only a *skew field*. In spite of that limitation, quaternions have become increasingly important in modern physics. Just like complex numbers, quaternions can also be thought of as elements

$$q = a \cdot 1 + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$$

in $\mathbb{R}^4$ with a special product.

Exercise A. a. Prove Pascal's rule:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$$

Hint: Write out the right-hand side in terms of factorials, put over a common denominator and simplify.

b. Use Pascal's rule and mathematical induction on n to prove the binomial formula for complex numbers:

$$(z+w)^n = z^n + \binom{n}{1} z^{n-1} w + \cdots + \binom{n}{k} z^{n-k} w^k + \cdots + \binom{n}{n-1} z w^{n-1} + w^n.$$

In proving this formula, it might be helpful to first write it in summation notation:

$$(z+w)^n = \sum_{k=0}^{n} \binom{n}{k} z^{n-k} w^k.$$

3 Polar coordinates

Complex addition is just vector addition, but complex multiplication is a little harder to visualize. To fully understand complex multiplication, it is convenient to use polar coordinates in the complex plane:

$$x = r\cos\theta, \quad y = r\sin\theta.$$

We can then write

$$z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta).$$

We call r the *modulus* and θ the *argument* of the complex number z; note that the argument $\theta = \arg(z)$ is only defined up to the addition of an integer multiple of 2π. We let $\operatorname{Arg}(z)$ denote the unique value of $\arg(z)$ which lies in the interval $(-\pi, \pi]$, and call it the *principal value* of the argument.

We often write

$$\cos\theta + i\sin\theta = e^{i\theta}, \tag{3}$$

and often call this *Euler's identity*. To give a rigorous version of this identity, we would need to investigate convergence of power series, which indeed is done in Math 117 or in Chapter 5 of [5]. However, the reader has probably seen power series already in calculus courses, sufficient to at least motivate the expression for $e^{i\theta}$. To see how Euler's identity arises, we start with the McClaurin expansions for e^x, $\sin x$ and $\cos x$:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 \cdots + \cdots + \frac{1}{n!}x^n + \cdots,$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots,$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots.$$

Then substituting $i\theta$ for x and assuming that the power series converge, we obtain

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{i^2\theta^2}{2!}x^2 + \frac{i^3\theta^3}{3!}x^3 + \frac{i^4\theta^4}{4!}x^4 \cdots + \cdots \\ &= \left(1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \cdots\right) + i\left(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \cdots\right) \\ &= \cos\theta + i\sin\theta, \end{aligned}$$

which is exactly the identity we needed to establish. For the time being, the reader can simply think of (3) as defining $e^{i\theta}$. Once we have Euler's identity at our disposal, we can write the polar form of a complex number as

$$z = re^{i\theta}.$$

It is the polar form of complex numbers which makes complex multiplication easy to visualize. Indeed, if

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2),$$

then

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)] \\ &= r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)), \end{aligned}$$

where we have used the familiar formulae for the cosine and sine of the sum of two angles. Thus to multiply two complex numbers together, we multiply the moduli and add the arguments, which is expressed in terms of Euler's identity as

$$(r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = (r_1 r_2)e^{i(\theta_1+\theta_2)}.$$

An important special case of this calculation is

$$e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}. \tag{4}$$

This interpretation of complex multiplication makes it relatively easy to calculate square roots; indeed,

$$\sqrt{re^{i\theta}} = \pm\sqrt{r}e^{i\theta/2}.$$

For example,

$$\sqrt{i} = \sqrt{e^{i\pi/2}} = \pm e^{i\pi/4} = \pm(\cos(\pi/4) + i\sin(\pi/4)) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i.$$

More generally, suppose that we want to calculate the m-th root of a complex number z. There are actually m such m-th roots. Indeed, the m-th roots of unity are simply

$$\omega_m = e^{2\pi i/m} = \cos\left(\frac{2\pi}{m}\right) + i\sin\left(\frac{2\pi}{m}\right)$$

and all of its powers. We can visualize the distinct m-th roots of unity,

$$\omega_m, \omega_m^2, \ldots, \omega_m^{m-1}, 1,$$

as equally spaced points around the unit circle. To find the m-th roots of a real number, we find the positive real m-th root and multiply it by all the m-th roots of unity.

For example, the three cube roots of unity are just

$$\omega_3 = \cos\left(\frac{2\pi}{m}\right) + i\sin\left(\frac{2\pi}{m}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i. \quad \omega_3^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i, \quad \omega_3^3 = 1,$$

and so the three complex cube roots of 8 are just

$$2\omega_3 = -1 + \sqrt{3}i, \quad 2\omega_3^2 = -1 - \sqrt{3}i, \quad 2\omega_3^2 = 2.$$

4 The complex exponential

The main goal of Math 122A is to study complex-valued functions of a complex variable z. One of the most important of these functions is the complex exponential $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y) = e^x \cos y + ie^x \sin y,$$

where we have used Euler's identity (3) to express e^{iy} in terms of $\cos y$ and $\sin y$. It is often convenient to write

$$f(z) = f(x, y) = u(x, y) + iv(x, y),$$

where

$$u(x, y) = e^x \cos y, \quad v(x, y) = e^x \sin y$$

are smooth functions of x and y, called the *real* and *imaginary parts* of the complex-valued function f.

Note that since

$$f(z + 2\pi in) = f(z), \quad \text{for } n \in \mathbb{Z},$$

the function f is not one-to-one. It is also not onto, because there is no $z \in \mathbb{C}$ such that $f(z) = 0$. Nevertheless, we can define a partial inverse function

$$\text{Log} : \mathbb{C} - \{x \in \mathbb{R} : x \leq 0\} \to \mathbb{C} \quad \text{by} \quad \text{Log}(z) = \text{Log}|z| + i\text{Arg}(z),$$

where the logarithm on the right is the usual natural logarithm of a real number. Note that the function Log is defined everywhere except on the negative x-axis. If we tried to extend it over the negative x-axis, we would have to introduce a jump discontinuity.

From (4), we easily conclude that the exponential function has an important special property

$$e^{z_1+z_2} = e^{z_1}e^{z_2}.$$

The exponential function and its inverse are examples which will recur throughout the course.

Of course, when we let $z = i\theta$, where θ is real, the complex exponential just reduces to Euler's identity (3). The conjugate of Euler's identity is

$$\cos\theta - i\sin\theta = e^{-i\theta}, \tag{5}$$

and we can solve (3) and (5) for cosine and sine, obtaining

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}. \tag{6}$$

More generally, we can replace θ by an arbitrary complex number:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

Thus over the complex numbers, the trigonometric functions can be defined in terms of the exponential. This fact can often be exploited to advantage.

For example, suppose that we want to calculate the integral

$$\int_0^{2\pi} \sin^4\theta d\theta.$$

Although this might appear to be difficult to do directly, it is easy if we substitute from (6) and expand using the binomial theorem:

$$\sin^4\theta = \frac{1}{2i}^4 \left(e^{i\theta} - e^{-i\theta}\right)^4$$

$$= \frac{1}{2^4}\left(e^{4i\theta} - 4e^{2i\theta} + 6 - 4e^{-2i\theta} + e^{4i\theta}\right)$$

$$= \frac{1}{2^4}\left(2\cos(4\theta) - 8\cos(2\theta) + 6\right).$$

Since $\cos(2\theta)$ and $\cos(4\theta)$ integrate to zero, we see that

$$\int_0^{2\pi} \sin^4\theta d\theta = \int_0^{2\pi} \frac{3}{8} d\theta = \frac{3\pi}{4}.$$

Conversely, we can express $\cos(n\theta)$ and $\sin(n\theta)$ in terms of powers of $\cos\theta$ and $\sin\theta$ by means of *de Moivre's formula*:

$$\cos(n\theta) + i\sin(n\theta) = (\cos\theta + i\sin\theta)^n.$$

For example, suppose that we want to $cos(4\theta)$ in terms of powers of $\cos\theta$ and $\sin\theta$. We can write

$$\cos(4\theta) + i\sin(4\theta) = (\cos\theta + i\sin\theta)^4,$$

expand the right-hand side using the binomial formula, and compare the real parts of the two sides. Since

$$\cos(4\theta) + i\sin(4\theta)$$
$$= \cos^4\theta + 4i\cos\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos^3\theta\sin\theta + \sin^4\theta,$$

we can take the real parts of both sides, and conclude that

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta.$$

5 Subsets of the complex plane

We have seen that the complex plane $\mathbb{C}$ is just the Euclidean plan $\mathbb{R}^2$ together with two operations which make it into a field, vector addition and complex multiplication. It is therefore not too surprising that much of the terminology from $\mathbb{R}^n$ (as described in [4] for example) carries over directly to the complex plane.

In particular, the *distance* between two points $z_0 = x_0+iy_0$ and $z_1 = x_1+iy_1$ within $\mathbb{C}$ is given by the Pythagorean formula

$$d(z_0, z_1) = |z_0 - z_1| = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2}.$$

This distance function satisfies all of the usual axioms which make $\mathbb{C}$ into what is called a metric space:

1. $d(z_0, z_1) \geq 0$ and $d(z_0, z_1) = 0 \Leftrightarrow z_0 = z_1$,
2. $d(z_0, z_1) = d(z_1, z_2)$, and
3. $d(z_0, z_2) \leq d(z_0, z_1) + d(z_1, z_2)$,

whenever z_0, z_1 and z_2 are elements of $\mathbb{C}$.

It is possible to describe many interesting subsets of $\mathbb{C}$ in terms of the distance function. Thus for example,

$$S = \{z \in \mathbb{C} : |z - (2 + i)| = 4\} = \{z \in \mathbb{C} : d(z, (2 + i)) = 4\} \tag{7}$$

is just the set of points which are four units away from $2 + i$, which is of course the circle of radius 4 centered at $2 + i$. Similarly,

$$S = \{z \in \mathbb{C} : |z - (3 + i)| = |z - (5 + 2i)|\} \tag{8}$$

is just the set of points which are equidistant from the two points $3 + i$ and $5 + 2i$, which is the straight line which bisects the line segment from $3 + i$ to $5 + 2i$.

There is much terminology associated with distance functions from metric spaces which has become part of the fabric of contemporary mathematics. We will need this terminology for our study of complex analysis.

Definition. If ε is a positive number, the *open disk* of radius ε about the point $z_0 \in \mathbb{C}$ is the subset

$$N(z_0; \varepsilon) = \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}.$$

We sometimes also call this an ε-neighborhood about z_0.

Definition. A subset $U \subseteq \mathbb{C}$ is said to be *open* if whenever $z \in U$, there is an open disk $N(z; \varepsilon)$ of some positive radius ε about z such that $N(z; \varepsilon) \subseteq U$.

Roughly speaking, a subset $U \subseteq \mathbb{C}$ is open if whenever $z \in U$, any point sufficiently close to z is also in U. Thus, for example, the set $\{z \in \mathbb{C} : |z| < 1\}$ is open while the set $\{z \in \mathbb{C} : |z| \leq 1\}$ is not.

Proposition 1. *The empty set $\emptyset$ and the whole space $\mathbb{C}$ are open subsets of $\mathbb{C}$. The union of an arbitrary collection of open sets is open. The intersection of a finite collection of open sets is open.*

For completeness, we include a proof, although some may prefer to accept the theorem on faith. In particular, we leave it to the reader to check that the empty set $\emptyset$ and the whole space $\mathbb{C}$ are open subsets of $\mathbb{C}$. To prove that an arbitrary union of open sets is open, we could suppose that $\{U_\alpha : \alpha \in A\}$ is a collection of open sets and that

$$U = \bigcup \{U_\alpha : \alpha \in A\}.$$

If $z \in U$, then $z \in U_\alpha$ for some $\alpha \in A$. Hence there is an $\varepsilon > 0$ such that $N(z; \varepsilon) \subseteq U_\alpha$. But then $N(z; \varepsilon) \subseteq U$, and this shows that U is open.

On the other hand, suppose that $\{U_1, \ldots, U_m\}$ is a finite collection of open sets and that

$$U = U_1 \cap \cdots \cap U_m.$$

If $z \in U$, then $z \in U_i$ for every i, $1 \leq i \leq m$. Hence for each i, $1 \leq i \leq m$, there exists $\varepsilon_i > 0$ such that $N(z; \varepsilon_i) \subseteq U_i$. Let

$$\varepsilon = \min(\varepsilon_1, \ldots, \varepsilon_m),$$

and note that $\varepsilon > 0$ since the minimum of a finite number of positive numbers is positive. Then $N(z; \varepsilon) \subseteq U_i$ for every i, $1 \leq i \leq m$. Hence $N(z; \varepsilon) \subseteq U$ and the finite intersection U is open, finishing our proof.

Definition. A subset $S \subseteq \mathbb{C}$ is said to be *closed* if $\mathbb{C} - S$ is open.

For example, the circle (7) and the line (8) are closed subsets of $\mathbb{C}$ which are not open. Any finite subset of $\mathbb{C}$ is closed. By an argument similar to that of the Proposition, one can show that the empty set $\emptyset$ and the whole space $\mathbb{C}$ are closed, the intersection of an arbitrary collection of closed sets is closed, and the union of a finite collection of closed sets is closed.

Particular types of open and closed sets are also important.

Definition. An open subset $U \subseteq \mathbb{C}$ is said to be *connected* if whenever $z_0, z_1 \in U$, there is a smooth path from z_0 to z_1 which lies entirely within U.

In this definition, one can replace "smooth path" by "polygonal path," and in fact this is done by Saff and Snider in §1.6 of [5]. Connected open sets are quite important within complex analysis, so Saff and Snider have a special name such set; connected open sets are called *domains*.

Proposition 2. *Suppose that U is a connected open subset of $\mathbb{C}$. If $u : U \to \mathbb{R}$ is a function with continuous partial derivatives such that*

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0, \tag{9}$$

then u is constant.

A proof, using ideas from calculus, follows from the chain rule. Indeed, if $z_0, z_1 \in U$ and $\gamma : [a, b] \to U$ is a smooth path such that $\gamma(a) = z_0$ and $\gamma(b) = z_1$, say

$$\gamma(t) = x(t) + iy(t), \quad \text{for } t \in [a, b],$$

then

$$u(z_1) - u(z_0) = \int_a^b \frac{d}{dt}(u \circ \gamma)(t)dt = \int_a^b \left(\frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \right) dt = 0,$$

so $u(z_1) = u(z_0)$. Thus u has the same value at any two points of U, which is exactly what we needed to prove.

Example. Suppose that $U = \{z = x + iy \in \mathbb{C} : x \neq 0\}$. Then the function $u : U \to \mathbb{R}$ defined by

$$u(x + iy) = \begin{cases} 3, & \text{if } x < 0, \\ 5, & \text{if } x > 0, \end{cases}$$

satisfies (9) yet is not constant. Of course, in this case, U is open but not connected.

Among the most important closed sets are those which are bounded.

Definition. A subset $S \subseteq \mathbb{C}$ is said to be *bounded* if there is a positive real number R such that

$$z \in S \quad \Rightarrow \quad |z| \leq R.$$

A subset $K \subseteq \mathbb{C}$ is *compact* if it is both closed and bounded

One of the key theorems from real analysis ([4], Corollary 24.10) states that a continuous real-valued function defined on a compact set $K \subseteq \mathbb{C}$ must achieve its maximum and minimum values at some points of K.

6 The Riemann sphere

The *Riemann sphere* is the space obtained from the complex plane $\mathbb{C}$ by adding a point at infinity, which we denote by ∞. It is most often visualized, however, via *stereographic projection* from the (x_1, x_2)-plane to the unit sphere

$$S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$$

If $N = (0, 0, 1)$ is the north pole on S^2, stereographic projection is a one-to-one onto function

$$\Phi : \mathbb{C} \longrightarrow S^2 - \{N\},$$

where $\mathbb{C}$ is thought of as the (x_1, x_2)-plane

We can follow the calculations in §1.6 of [5] to define the stereographic projection. We consider the line L containing $N = (0, 0, 1)$ on S^2 and the point $(x, y, 0)$ in the plane $x_3 = 0$. This line L can be parametrized by

$$\gamma : \mathbb{R} \to \mathbb{R}^3 \quad \text{where} \quad \gamma(t) = (x_1(t), x_2(t), x_3(t)) = t(x, y, 0) + (1 - t)(0, 0, 1),$$

or alternatively, we can write this as

$$x_1(t) = tx, \quad x_2(t) = ty, \quad x_3(t) = 1 - t. \tag{10}$$

There is a unique nonzero value for t such that $\gamma(t)$ lie on $S^2 - \{N\}$, and it occurs when

$$1 = x_1^2 + x_2^2 + x_3^2 = t^2x^2 + t^2y^2 + (1 - t)^2.$$

We can expand and solve for t:

$$1 = t^2(x^2 + y^2) + 1 - 2t + t^2 = t^2(x^2 + y^2 + 1) - 2t + 1,$$
$$2t = t^2(x^2 + y^2 + 1), \quad 2 = t(x^2 + y^2 + 1), \quad t = \frac{2}{x^2 + y^2 + 1}.$$

Substitution into (10) then yields the point on S^2 which corresponds to the point $(x, y, 0)$, corresponding to $x + iy \in \mathbb{C}$:

$$x_1 = \frac{2x}{x^2 + y^2 + 1}, \quad x_2 = \frac{2y}{x^2 + y^2 + 1}, \quad x_3 = 1 - \frac{2}{x^2 + y^2 + 1}.$$

We can then simplify this to

$$x_1 = \frac{2x}{x^2 + y^2 + 1}, \quad x_2 = \frac{2y}{x^2 + y^2 + 1}, \quad x_3 = \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}, \tag{11}$$

which gives an explicit formula for stereographic projection

$$\Phi : \mathbb{C} \longrightarrow S^2 - \{N\}, \quad \Phi(z) = \left(\frac{2\mathrm{Re}(z)}{|z|^2 + 1}, \frac{2\mathrm{Im}(z)}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1}\right),$$

where $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ are the real and imaginary parts of z.

To see that Φ is one-to-one and onto, we can construct an explicit inverse. Indeed, eliminating t from equations (10) gives first $t = 1 - x_3$ and then

$$x = \frac{x_1}{t} = \frac{x_1}{1 - x_3}, \quad y = \frac{x_2}{t} = \frac{x_2}{1 - x_3}.$$

Thus we can define a function

$$\phi : S^2 - \{N\} \to \mathbb{C} \quad \text{by} \quad \phi(x_1, x_2, x_3) = \frac{x_1 + ix_2}{1 - x_3}.$$

which is exactly the inverse of Φ. Note that the inverse map ϕ is well-behaved except when $x_3 = 1$, that is, it is well-behaved except at the north pole N on S^2.

These explicit formulae may seem confusing at first. The important point to note is that as the modulus of z gets larger and larger, $\Phi(z)$ approaches the north pole on S^2. Thus if we think of using Φ to identify points of $\mathbb{C}$ with points on $S^2 - \{N\}$, then the north pole N should be identified with a point at infinity. Indeed we might think of extending Φ to a map

$$\tilde{\Phi} : \mathbb{C} \cup \{\infty\} \longrightarrow S^2.$$

This idea of adding a point at infinity to the complex plane, thereby obtaining what is sometimes called *the extended complex plane* or the *one-point compactification* of $\mathbb{C}$, has turned out to be extremely useful in understanding functions of a complex variable.

Example. Supppose that $U = \mathbb{C} - \{3\}$ and

$$f : U \to \mathbb{C} \quad \text{by} \quad f(z) = \frac{1}{z - 3}.$$

Since $f(z)$ gets larger and larger as z approaches 3, it is often useful to extend f to a map

$$\hat{f} : \mathbb{C} \to \mathbb{C} \cup \{\infty\} \quad \text{so that} \quad f(z) = \begin{cases} 1/(z-3), & \text{if } z \neq 3, \\ \infty, & \text{if } z = 3. \end{cases}$$

Unfortunately, there is no way of defining addition and multiplication with ∞ so that $\mathbb{C} \cup \{\infty\}$ satisfies the field axioms.

We can think of ϕ as defining a coordinate z on S^2, which is well-behaved everywhere except at the north pole N. But sometimes we want a coordinate w that might be well-behaved near ∞. We might try to take

$$w = \frac{1}{z}$$

as such a coordinate. Can we think of this also as a complex coordinate on part of S^2? We find that

$$\begin{aligned} z &= \frac{x_1 + ix_2}{1 - x_3} \\ &\Rightarrow \quad w = \frac{1}{z} = \frac{1 - x_3}{x_1 + ix_2} = \frac{(1 - x_3)(x_1 - ix_2)}{x_1^2 + x_2^2} \\ &= \frac{(1 - x_3)(x_1 - ix_2)}{1 - x_3^2} = \frac{x_1 - ix_2}{1 + x_3} \end{aligned}$$

Thus if we let S denote the south pole $(0, 0, -1)$, we can define

$$\psi : S^2 - \{S\} \to \mathbb{C} \quad \text{by} \quad w = \psi(x_1, x_2, x_3) = \frac{x_1 - ix_2}{1 + x_3}.$$

It tuns out that conjugation followed by the inverse of ψ is then stereographic projection with the north pole replaced by the south pole.

These constructions give two complex coordinate systems on S^2,

$$\phi : S^2 - \{N\} \to \mathbb{C} \quad \text{and} \quad \psi : S^2 - \{S\} \to \mathbb{C}$$

which are related by

$$\psi \circ \phi^{-1}(z) = w = \frac{1}{z}.$$

These complex coordinate systems make S^2 into what is called a *Riemann surface*.

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