Complex Varables
Lecture Notes for Math 122A

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Contents

Chapter 1

Complex Numbers

These notes are intended to supplement the text, *Fundamentals of complex analysis*, by Saff and Snider [10]. Other often-used references for the theory of analytic functions of a complex variable are the alternate text by Brown and Churchill [2], and the more advanced classic by Ahlfors [1]. For a history of the development of complex numbers, we recommend relevant chapters of [5].

If we were to develop real and complex analysis from the foundation up, we would start with set theory (as studied in Math 8). Using sets, we would build up successively the natural numbers, the integers, the rational numbers and the real numbers.

We would start by defining the set of *natural numbers*

$$\mathbb{N} = \{1, 2, 3, \ldots\} \quad \text{and} \quad \omega = \{0\} \cup \mathbb{N} = \{0, 1, 2, 3, \ldots\},$$

together with the usual operations of addition and multiplication in terms of sets. We could then define an equivalence relation $\sim$ on the Cartesian product $\omega \times \omega$ by

$$(m, n) \sim (q, r) \quad \Leftrightarrow \quad m + r = q + n.$$

The equivalence class $[m, n]$ of the pair $(m, n) \in \omega \times \omega$ would be thought of as the difference $m - n$, and the set of equivalence classes

$$\mathbb{Z} = \{\ldots, -2 = [0, 2], -1 = [0, 1], 0 = [0, 0], 1 = [1, 0], 2 = [2, 0], \ldots\}$$

would then be regarded as the set of *integers*. We would next define the usual addition and multiplication on $\mathbb{Z}$ and show that these operations satisfy the familiar properties. The advantage of $\mathbb{Z}$ over $\mathbb{N}$ is that subtraction is always defined.

Next, we would define an equivalence relation $\sim$ on the Cartesian product $\mathbb{Z} \times (\mathbb{Z} - \{0\})$ by

$$(m, n) \sim (q, r) \quad \Leftrightarrow \quad mr = qn,$$

and think of the equivalence class $[m, n]$ are representing the fraction m/n. The set of all such fractions is known as the set $\mathbb{Q}$ of rational numbers. We could then define addition and multiplication by

$$\frac{m}{n} + \frac{q}{r} = \frac{mr + qn}{nr}, \quad \frac{m}{n} \cdot \frac{q}{r} = \frac{mq}{nr},$$

and establish all the usual rules of arithmetic with rational numbers, familiar from grade school, including now division. A complete construction would be long and time-consuming, and you might wonder whether it isn't a bit pedantic to carry this out with so much rigor. But it is important to understand that all of the familiar rules of arithmetic for rational numbers can in fact be established by deduction from the axioms of set theory.

The last stage is developing the real numbers $\mathbb{R}$, which can be thought of as limits of sequences of rational numbers. For example, the number π is the limit of the sequence

$$(3, 3.1, 3.14, 3.141, 3.1415, 3.14159, 3.141592, \ldots, 3.14159265358979, \ldots).$$

It is precisely the notion of defining the limit of such a sequence which is the major difficulty in developing real analysis. It would take a long time just to define the real numbers in this manner, so for a first treatment of real analysis, most authors take a shortcut, and formulate a collection of axioms which characterize the real numbers. One often assumes these axioms as the starting point of real analysis, rather than just the axioms of set theory. (Since one does want to use the properties of sets in discussing real numbers, a full formal development of analysis in this shortened form would require both the axioms of set theory and the axioms of real numbers. On the other hand, many authors, just use set theory as a basic language whose basic properties are intuitively clear; this is more or less the way mathematicians thought about set theory prior to its study by Georg Cantor (1845-1918) and its later axiomatization.)

The rational numbers and the real numbers both satisfy the field axioms that we next describe.

1.1 Field axioms

Definition. A *field* is a set F together with two operations (functions)

$$f : F \times F \to F, \qquad f(x, y) = x + y$$

and

$$g : F \times F \to F, \qquad g(x, y) = xy,$$

called addition and multiplication, respectively, which satisfy the following field axioms:

- F1. addition is commutative: $x + y = y + x$, for all $x, y \in F$.
- F2. addition is associative: $(x + y) + z = x + (y + z)$, for all $x, y, z \in F$.

- F3. existence of additive identity: there is a unique element $0 \in F$ such that $x + 0 = x$, for all $x \in F$.

- F4. existence of additive inverses: if $x \in F$, there is a unique element $-x \in F$ such that $x + (-x) = 0$.

- F5. multiplication is commutative: $xy = yx$, for all $x, y \in F$.

- F6. multiplication is associative: $(xy)z = x(yz)$, for all $x, y, z \in F$.

- F7. existence of multliplicative identity: there is a unique element $1 \in F$ such that $1 \neq 0$ and $x1 = x$, for all $x \in F$.

- F8. existence of multliplicative inverses: if $x \in F$ and $x \neq 0$, there is a unique element $(1/x) \in F$ such that $x \cdot (1/x) = 1$.

- F9. distributivity: $x(y + z) = xy + xz$, for all $x, y, z \in F$.

Note the similarity between axioms F1-F4 and axioms F5-F8. In the language of algebra, axioms F1-F4 state that F with the addition operation f is an *abelian group*. (The group axioms are studied further in the first part of abstract algebra, which is devoted to group theory.) Axioms F5-F8 state that $F - \{0\}$ with the multiplication operation g is also an abelian group. Axiom F9 ties the two field operations together.

Among the most important examples of fields are the set of rational numbers $\mathbb{Q}$ and the set of real numbers $\mathbb{R}$. In both cases we take f and g to be the usual addition and multiplication operations. On the other hand, the set of integers $\mathbb{Z}$ with the usual addition and multiplication is NOT a field, because integers do not always have multiplicative inverses.

The field of reals $\mathbb{R}$ is much larger than the field $\mathbb{Q}$ of rationals. Indeed, as you have most likely seen in Math 8, Georg Cantor proved that the field $\mathbb{Q}$ of rational numbers is countable, that is, in one-to-one correspondence with $\mathbb{N}$, while the field $\mathbb{R}$ is uncountable.

Another example. We can define a field $\mathbb{Z}/p\mathbb{Z}$, where p is a prime ≥ 2, which consists of the elements $\{0, 1, 2, \ldots, p-1\}$. In this case, we define addition or multiplication by first forming the sum or product in the usual sense and then taking the remainder after division by p, so as to arrive back in the set $\{0, 1, 2, \ldots, p-1\}$. This is often referred to as mod p addition and multiplication. Thus for example,

$$\mathbb{Z}/5\mathbb{Z} = \{0, 1, 2, 3, 4\},$$

and within $\mathbb{Z}/5\mathbb{Z}$,

$$3 + 4 = 7 \bmod 5 = 2, \qquad 3 \cdot 4 = 12 \bmod 5 = 2.$$

One can use theorems on prime factorization to show that $\mathbb{Z}/p\mathbb{Z}$ satisfies all the field axioms, and is therefore a field which contains only finitely many elements.

On the other hand, if n is not a prime, then $\mathbb{Z}/n\mathbb{Z}$ with mod n addition and multiplication is NOT a field. Indeed, in $\mathbb{Z}/4\mathbb{Z}$,

$$2 + 2 = 4 \bmod 4 = 0,$$

so 2 does not have a multiplicative inverse in $\mathbb{Z}/4\mathbb{Z}$, contradicting Axiom F8, so $\mathbb{Z}/4\mathbb{Z}$ is not a field.

1.2 Complex numbers

Unfortunately, it is not possible to take the square roots of a negative real number and get a real number as a result. This defect makes it impossible to find solutions to polynomial equations like

$$x^2 + 1 = 0$$

when using just real numbers. In order to remedy this problem, we introduce the complex numbers $\mathbb{C}$. There are two common ways of doing this:

Method I. We can utilize the theory of matrices, and regard the space $\mathbb{C}$ of complex numbers to be the set of 2×2 matrices of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

where a and b are real numbers. One can check that the sum or product of two elements of $\mathbb{C}$ is again an element of $\mathbb{C}$. Although matrix do not commute in general, it is the case that

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} c & -d \\ d & c \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

for any choice of a, b, c and d, as you can easily verify by direct multiplication. We often use the notation

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

so that

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a + bi.$$

The set $\mathbb{C}$ of matrices we have described, together with the operations of matrix addition and multiplication, satisfies the field axioms, and we can call it the field of complex numbers.

Method II. We can also think of the space $\mathbb{C}$ of complex numbers as the space $\mathbb{R}^2$ of ordered pairs of real numbers (a, b) using vector addition for addition, with the additional structure of a multiplication defined by the formula

$$(a, b)(c, d) = (ac - bd, ad + bc). \tag{1.1}$$

Alternatively, we can set

$$1 = (1, 0), \quad i = (0, 1), \quad \text{so} \quad (a, b) = a + bi,$$

and the formula (1.1) then shows that $i^2 = -1$. Once again, the set $\mathbb{C}$ of complex numbers is a field under the addition and multiplication operations that we have defined. Thus all of the usual rules of arithmetic (such as the associative, commutative and distributive laws) can be applied to complex numbers.

The second approach is the one adopted by the text [10], and suggests an important way of visualizing complex numbers. A complex number

$$z = x + iy$$

can be thought of as representing a point in the (x, y)-plane. We say that x is the *real part* of z, while y is the *imaginary part*, and we write

$$x = \mathrm{Re}(z), \quad y = \mathrm{Im}(z).$$

Using the Pythagorean law, we can define the length or *modulus* of the complex number $z = x + iy$ by

$$|z| = \sqrt{x^2 + y^2}.$$

Of the main operations on complex numbers, only division might provide a challenge for calculation when starting out. If $a + ib$ is a complex number, its *conjugate* is

$$\overline{a + ib} = a - ib.$$

The division of complex numbers is then obtained by multiplying both numerator and denominator by the conjugate of the denominator:

$$\frac{3 + 5i}{2 + 3i} = \frac{3 + 5i}{2 + 3i}\frac{2 - 3i}{2 - 3i} = \frac{21 + i}{13}.$$

The complex numbers provide an important extension of the real numbers, because within the complex numbers, one can always solve quadratic equations. Recall that if $a, b, c \in \mathbb{R}$, the roots of the quadratic equations

$$az^2 + bz + c = 0 \qquad \text{are} \qquad z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{1.2}$$

The solutions can always be written as complex numbers, because we can always find a square root of $b^2 - 4ac$, even if it is negative.

Indeed, it is with the quadratic formula (1.2) that students usually encounter complex numbers for the first time. Although they appear strange at first, it soon becomes apparent that the complex numbers pay for themselves many times over in many problems in which one needs to find a solution within the real numbers to a problem expressed in terms of the real numbers.

Of course, one might try to find a similar formula for zeros of the cubic

$$az^3 + bz^2 + cz + d = 0.$$

Finding such a formula was one of the successes of the Renaissance mathematicians in Italy, a solution in terms of radicals being found by Tartaglia and Cardano in 1545. Although these Italian mathematicians did not use complex numbers, their cumbersome methods are far easier to understand with complex numbers. An even more complicated formula was found for zeros of quartics. Évariste Galois (1811-32) was able to show that there is no formula in terms of radicals (square roots, cube roots and so forth) for zeros to the most general quintic polynomials.

Nevertheless, a far-reaching existence theorem on roots to polynomials is usually attributed to Carl Friedrich Gauss (1777-1855):

Fundamental Theorem of Algebra. *Every nonconstant polynomial with complex coefficients has at least one complex zero.*

When the degree of the polynomial is large, one is usually forced to use numerical methods to find approximations to the zeros. The Fundamental Theorem provides one of the main reasons for the importance of complex numbers. We will give a proof of this important result later in the course.

Quaternions. One might wonder whether it is possible to extend the notion of complex numbers yet again to a larger field. This was tried by Sir William Rowen Hamilton (1805-65) who developed the quaternions as a result; see [5], pages 776-782. In modern notation, we would define the space $\mathbb{H}$ of *quaternions* to be the set of 2×2 matrices of the form

$$\begin{pmatrix} z & -\bar{w} \\ w & z \end{pmatrix},$$

where z and w are complex numbers with conjugates $\bar{z}$ and $\bar{w}$. Once again, one can check that the sum or product of two elements of $\mathbb{H}$ is again in $\mathbb{H}$. The operations of matrix addition and multiplication satisfy all of the field operations except for commutativity of multiplication F5. Indeed, one can check that if

$$\mathbf{i} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

then

$$\mathbf{ij} = \mathbf{k}, \quad \text{while} \quad \mathbf{ji} = -\mathbf{k},$$

in analogy with the cross product. Thus the sets of quaternions $\mathbb{H}$ is not quite a field, but only a *skew field*. In spite of that limitation, quaternions have become increasingly important in modern physics. Just like complex numbers, quaternions can also be thought of as elements

$$q = a \cdot 1 + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$$

in $\mathbb{R}^4$ with a special product.

Exercise A. a. Prove Pascal's rule:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$$

Hint: Write out the right-hand side in terms of factorials, put over a common denominator and simplify.

b. Use Pascal's rule and mathematical induction on n to prove the binomial formula for complex numbers:

$$(z+w)^n = z^n + \binom{n}{1} z^{n-1} w + \cdots + \binom{n}{k} z^{n-k} w^k + \cdots + \binom{n}{n-1} z w^{n-1} + w^n.$$

In proving this formula, it might be helpful to first write it in summation notation:

$$(z+w)^n = \sum_{k=0}^{n} \binom{n}{k} z^{n-k} w^k.$$

1.3 Polar coordinates

Complex addition is just vector addition, but complex multiplication is a little harder to visualize. To fully understand complex multiplication, it is convenient to use polar coordinates in the complex plane:

$$x = r\cos\theta, \quad y = r\sin\theta.$$

We can then write

$$z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta).$$

We call r the *modulus* and θ the *argument* of the complex number z; note that the argument $\theta = \arg(z)$ is only defined up to the addition of an integer multiple of 2π. We let $\mathrm{Arg}(z)$ denote the unique value of $\arg(z)$ which lies in the interval $(-\pi, \pi]$, and call it the *principal value* of the argument.

We often write

$$\cos\theta + i\sin\theta = e^{i\theta}, \tag{1.3}$$

and call this *Euler's identity*. To give a rigorous version of this identity, we would need to investigate convergence of power series, which indeed is done in Math 117 or in Chapter 5 of [10]. However, the reader has probably seen power series already in calculus courses, sufficient to at least motivate the expression for $e^{i\theta}$. To see how Euler's identity arises, we start with the McClaurin expansions for e^x, $\sin x$ and $\cos x$:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 \cdots + \cdots + \frac{1}{n!}x^n + \cdots,$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots,$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots .$$

Then substituting $i\theta$ for x and assuming that the power series converge, we obtain

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{i^2\theta^2}{2!}x^2 + \frac{i^3\theta^3}{3!}x^3 + \frac{i^4\theta^4}{4!}x^4 \cdots + \cdots \\ &= \left(1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \cdots\right) + i\left(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \cdots\right) \\ &= \cos\theta + i\sin\theta, \end{aligned}$$

which is exactly what we wanted to establish. For the time being, the reader can simply think of (1.3) as defining $e^{i\theta}$. Once we have Euler's identity at our disposal, we can write the polar form of a complex number as

$$z = re^{i\theta}.$$

It is the polar form of complex numbers which makes complex multiplication easy to visualize. Indeed, if

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2),$$

then

$$\begin{aligned} z_1z_2 &= r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)] \\ &= r_1r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)), \end{aligned}$$

where we have used the familiar formulae for the cosine and sine of the sum of two angles. Thus to multiply two complex numbers together, we multiply the moduli and add the arguments, expressed in terms of Euler's identity as

$$(r_1e^{i\theta_1})(r_2e^{i\theta_2}) = (r_1r_2)e^{i(\theta_1+\theta_2)}.$$

An important special case of this calculation is

$$e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}. \tag{1.4}$$

This interpretation of complex multiplication makes it relatively easy to calculate square roots; indeed,

$$\sqrt{re^{i\theta}} = \pm\sqrt{r}e^{i\theta/2}.$$

For example,

$$\sqrt{i} = \sqrt{e^{i\pi/2}} = \pm e^{i\pi/4} = \pm(\cos(\pi/4) + i\sin(\pi/4)) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i.$$

More generally, suppose that we want to calculate the m-th root of a complex number z. There are actually m such m-th roots. Indeed, the m-th roots of unity are simply

$$\omega_m = e^{2\pi i/m} = \cos\left(\frac{2\pi}{m}\right) + i\sin\left(\frac{2\pi}{m}\right)$$

and all of its powers. We can visualize the distinct m-th roots of unity,

$$\omega_m, \omega_m^2, \ldots, \omega_m^{m-1}, 1,$$

as equally spaced points around the unit circle. To find the m-th roots of a real number, we find the positive real m-th root and multiply it by all the m-th roots of unity.

For example, the three cube roots of unity are just

$$\omega_3 = \cos\left(\frac{2\pi}{m}\right) + i\sin\left(\frac{2\pi}{m}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i. \quad \omega_3^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i, \quad \omega_3^3 = 1,$$

and so the three complex cube roots of 8 are just

$$2\omega_3 = -1 + \sqrt{3}i, \quad 2\omega_3^2 = -1 - \sqrt{3}i, \quad 2\omega_3^2 = 2.$$

1.4 The complex exponential

The main goal of Math 122A is to study complex-valued functions of a complex variable z. One of the most important of these functions is the complex exponential $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y) = e^x \cos y + ie^x \sin y,$$

where we have used Euler's identity (1.3) to express e^{iy} in terms of $\cos y$ and $\sin y$. It is often convenient to write

$$f(z) = f(x, y) = u(x, y) + iv(x, y),$$

where

$$u(x, y) = e^x \cos y, \quad v(x, y) = e^x \sin y$$

are smooth functions of x and y, called the *real* and *imaginary parts* of the complex-valued function f.

Note that since

$$f(z + 2\pi in) = f(z), \quad \text{for } n \in \mathbb{Z},$$

the function f is not one-to-one. It is also not onto, because there is no $z \in \mathbb{C}$ such that $f(z) = 0$. Nevertheless, we can define a partial inverse function

$$\mathrm{Log} : \mathbb{C} - \{x \in \mathbb{R} : x \leq 0\} \to \mathbb{C} \quad \text{by} \quad \mathrm{Log}(z) = \mathrm{Log}|z| + i\mathrm{Arg}(z),$$

where the logarithm on the right is the usual natural logarithm of a real number. Note that the function Log is defined everywhere except on the negative x-axis. If we tried to extend it over the negative x-axis, we would have to introduce a jump discontinuity.

From (1.4), we easily conclude that the exponential function has an important special property

$$e^{z_1+z_2} = e^{z_1}e^{z_2}.$$

The exponential function and its inverse are examples which will recur throughout the course.

Of course, when we let $z = i\theta$, where θ is real, the complex exponential just reduces to Euler's identity (1.3). The conjugate of Euler's identity is

$$\cos\theta - i\sin\theta = e^{-i\theta}, \tag{1.5}$$

and we can solve (1.3) and (1.5) for cosine and sine, obtaining

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}. \tag{1.6}$$

More generally, we can replace θ by an arbitrary complex number:

$$\cos z = \frac{e^{iz}+e^{-iz}}{2}, \quad \sin z = \frac{e^{iz}-e^{-iz}}{2i}.$$

Thus over the complex numbers, the trigonometric functions can be defined in terms of the exponential. This fact can often be exploited to advantage.

For example, suppose that we want to calculate the integral

$$\int_0^{2\pi} \sin^4\theta d\theta.$$

Although this might appear to be difficult to do directly, it is easy if we substitute from (1.6) and expand using the binomial theorem:

$$\begin{aligned}\sin^4\theta &= \left(\frac{1}{2i}\right)^4 \left(e^{i\theta}-e^{-i\theta}\right)^4 \\ &= \frac{1}{2^4}\left(e^{4i\theta}-4e^{2i\theta}+6-4e^{-2i\theta}+e^{4i\theta}\right) \\ &= \frac{1}{2^4}\left(2\cos(4\theta)-8\cos(2\theta)+6\right).\end{aligned}$$

Since $\cos(2\theta)$ and $\cos(4\theta)$ integrate to sines which cancel at the upper and lower limits, we see that

$$\int_0^{2\pi} \sin^4\theta d\theta = \int_0^{2\pi} \frac{3}{8} d\theta = \frac{3\pi}{4}.$$

Conversely, we can express $\cos(n\theta)$ and $\sin(n\theta)$ in terms of powers of $\cos\theta$ and $\sin\theta$ by means of *de Moivre's formula*:

$$\cos(n\theta) + i\sin(n\theta) = (\cos\theta + i\sin\theta)^n.$$

For example, suppose that we want to express $\cos(4\theta)$ in terms of powers of $\cos\theta$ and $\sin\theta$. We can write

$$\cos(4\theta) + i\sin(4\theta) = (\cos\theta + i\sin\theta)^4,$$

expand the right-hand side using the binomial formula, and compare the real parts of the two sides. Since

$$\begin{aligned}&\cos(4\theta) + i\sin(4\theta)\\ &\qquad = \cos^4\theta + 4i\cos\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos^3\theta\sin\theta + \sin^4\theta,\end{aligned}$$

we can take the real parts of both sides, and conclude that

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta.$$

1.5 Subsets of the complex plane

We have seen that the complex plane $\mathbb{C}$ is just the Euclidean plan $\mathbb{R}^2$ together with two operations which make it into a field, vector addition and complex multiplication. It is therefore not too surprising that much of the terminology from $\mathbb{R}^n$ (as described in [8] for example) carries over directly to the complex plane.

In particular, the *distance* between two points $z_0 = x_0 + iy_0$ and $z_1 = x_1 + iy_1$ within $\mathbb{C}$ is given by the Pythagorean formula

$$d(z_0, z_1) = |z_0 - z_1| = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2}.$$

This distance function satisfies all of the usual axioms which make $\mathbb{C}$ into what is called a metric space:

1. $d(z_0, z_1) \geq 0$ and $d(z_0, z_1) = 0 \Leftrightarrow z_0 = z_1$,
2. $d(z_0, z_1) = d(z_1, z_2)$, and
3. $d(z_0, z_2) \leq d(z_0, z_1) + d(z_1, z_2)$,

whenever z_0, z_1 and z_2 are elements of $\mathbb{C}$.

It is possible to describe many interesting subsets of $\mathbb{C}$ in terms of the distance function. Thus for example,

$$S = \{z \in \mathbb{C} : |z - (2+i)| = 4\} = \{z \in \mathbb{C} : d(z, (2+i)) = 4\} \qquad (1.7)$$

is just the set of points which are four units away from $2 + i$, which is of course the circle of radius 4 centered at $2 + i$. Similarly,

$$S = \{z \in \mathbb{C} : |z - (3+i)| = |z - (5+2i)|\} \qquad (1.8)$$

is just the set of points which are equidistant from the two points $3 + i$ and $5 + 2i$, which is the straight line which bisects the line segment from $3 + i$ to $5 + 2i$.

There is much terminology associated with distance functions from metric spaces which has become part of the fabric of contemporary mathematics. We will need this terminology for our study of complex analysis.

Definition. If ε is a positive number, the *open disk* of radius ε about the point $z_0 \in \mathbb{C}$ is the subset

$$N(z_0; \varepsilon) = \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}.$$

We sometimes also call this an ε-neighborhood about z_0.

Definition. A subset $U \subseteq \mathbb{C}$ is said to be *open* if whenever $z \in U$, there is an open disk $N(z; \varepsilon)$ of some positive radius ε about z such that $N(z; \varepsilon) \subseteq U$.

Roughly speaking, a subset $U \subseteq \mathbb{C}$ is open if whenever $z \in U$, any point sufficiently close to z is also in U. Thus, for example, the set $\{z \in \mathbb{C} : |z| < 1\}$ is open while the set $\{z \in \mathbb{C} : |z| \leq 1\}$ is not.

Proposition 1. *The empty set $\emptyset$ and the whole space $\mathbb{C}$ are open subsets of $\mathbb{C}$. The union of an arbitrary collection of open sets is open. The intersection of a finite collection of open sets is open.*

For completeness, we include a proof, although some may prefer to accept the theorem on faith. In particular, we leave it to the reader to check that the empty set $\emptyset$ and the whole space $\mathbb{C}$ are open subsets of $\mathbb{C}$. To prove that an arbitrary union of open sets is open, we could suppose that $\{U_\alpha : \alpha \in A\}$ is a collection of open sets and that

$$U = \bigcup \{U_\alpha : \alpha \in A\}.$$

If $z \in U$, then $z \in U_\alpha$ for some $\alpha \in A$. Hence there is an $\varepsilon > 0$ such that $N(z; \varepsilon) \subseteq U_\alpha$. But then $N(z; \varepsilon) \subseteq U$, and this shows that U is open.

On the other hand, suppose that $\{U_1, \ldots, U_m\}$ is a finite collection of open sets and that

$$U = U_1 \cap \cdots \cap U_m.$$

If $z \in U$, then $z \in U_i$ for every i, $1 \leq i \leq m$. Hence for each i, $1 \leq i \leq m$, there exists $\varepsilon_i > 0$ such that $N(z; \varepsilon_i) \subseteq U_i$. Let

$$\varepsilon = \min(\varepsilon_1, \ldots, \varepsilon_m),$$

and note that $\varepsilon > 0$ since the minimum of a finite number of positive numbers is positive. Then $N(z; \varepsilon) \subseteq U_i$ for every i, $1 \leq i \leq m$. Hence $N(z; \varepsilon) \subseteq U$ and the finite intersection U is open, finishing our proof.

Definition. A subset $S \subseteq \mathbb{C}$ is said to be *closed* if $\mathbb{C} - S$ is open.

For example, the circle (1.7) and the line (1.8) are closed subsets of $\mathbb{C}$ which are not open. Any finite subset of $\mathbb{C}$ is closed. By an argument similar to that of the Proposition, one can show that the empty set $\emptyset$ and the whole space $\mathbb{C}$

are closed, the intersection of an arbitrary collection of closed sets is closed, and the union of a finite collection of closed sets is closed.

Particular types of open and closed sets are also important.

Definition. An open subset $U \subseteq \mathbb{C}$ is said to be *connected* if whenever $z_0, z_1 \in U$, there is a smooth path from z_0 to z_1 which lies entirely within U.

In this definition, one can replace "smooth path" by "polygonal path," and in fact this is done by Saff and Snider in §1.6 of [10]. Connected open sets are quite important within complex analysis, so Saff and Snider have a special name such set; connected open sets are called *domains*.

Proposition 2. *Suppose that U is a connected open subset of $\mathbb{C}$. If $u : U \to \mathbb{R}$ is a function with continuous partial derivatives such that*

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0, \tag{1.9}$$

then u is constant.

A proof, using ideas from calculus, follows from the chain rule. Indeed, if $z_0, z_1 \in U$ and $\gamma : [a, b] \to U$ is a smooth path such that $\gamma(a) = z_0$ and $\gamma(b) = z_1$, say

$$\gamma(t) = x(t) + iy(t), \quad \text{for } t \in [a, b],$$

then

$$u(z_1) - u(z_0) = \int_a^b \frac{d}{dt}(u \circ \gamma)(t)dt = \int_a^b \left(\frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \right) dt = 0,$$

so $u(z_1) = u(z_0)$. Thus u has the same value at any two points of U, which is exactly what we needed to prove.

Example. Suppose that $U = \{z = x + iy \in \mathbb{C} : x \neq 0\}$. Then the function $u : U \to \mathbb{R}$ defined by

$$u(x + iy) = \begin{cases} 3, & \text{if } x < 0, \\ 5, & \text{if } x > 0, \end{cases}$$

satisfies (1.9) yet is not constant. Of course, in this case, U is open but not connected.

Among the most important closed sets are those which are bounded.

Definition. A subset $S \subseteq \mathbb{C}$ is said to be *bounded* if there is a positive real number R such that

$$z \in S \quad \Rightarrow \quad |z| \leq R.$$

A subset $K \subseteq \mathbb{C}$ is *compact* if it is both closed and bounded

One of the key theorems from real analysis ([8], Corollary 24.10) states that a continuous real-valued function defined on a compact set $K \subseteq \mathbb{C}$ must achieve its maximum and minimum values at some points of K.

1.6 The Riemann sphere

The *Riemann sphere* is the space obtained from the complex plane $\mathbb{C}$ by adding a point at infinity, which we denote by ∞. It is most often visualized, however, via *stereographic projection* from the (x_1, x_2)-plane to the unit sphere

$$S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}.$$

If $N = (0, 0, 1)$ is the north pole on S^2, stereographic projection is a one-to-one onto function

$$\Phi : \mathbb{C} \longrightarrow S^2 - \{N\},$$

where $\mathbb{C}$ is thought of as the (x_1, x_2)-plane

We can follow the calculations in §1.6 of [10] to define the stereographic projection. We consider the line L containing $N = (0, 0, 1)$ on S^2 and the point $(x, y, 0)$ in the plane $x_3 = 0$. This line L can be parametrized by

$$\gamma : \mathbb{R} \to \mathbb{R}^3 \quad \text{where} \quad \gamma(t) = (x_1(t), x_2(t), x_3(t)) = t(x, y, 0) + (1 - t)(0, 0, 1),$$

or alternatively, we can write this as

$$x_1(t) = tx, \quad x_2(t) = ty, \quad x_3(t) = 1 - t. \tag{1.10}$$

There is a unique nonzero value for t such that $\gamma(t)$ lie on $S^2 - \{N\}$, and it occurs when

$$1 = x_1^2 + x_2^2 + x_3^2 = t^2x^2 + t^2y^2 + (1 - t)^2.$$

We can expand and solve for t:

$$1 = t^2(x^2 + y^2) + 1 - 2t + t^2 = t^2(x^2 + y^2 + 1) - 2t + 1,$$
$$2t = t^2(x^2 + y^2 + 1), \quad 2 = t(x^2 + y^2 + 1), \quad t = \frac{2}{x^2 + y^2 + 1}.$$

Substitution into (1.10) then yields the point on S^2 which corresponds to the point $(x, y, 0)$, corresponding to $x + iy \in \mathbb{C}$:

$$x_1 = \frac{2x}{x^2 + y^2 + 1}, \quad x_2 = \frac{2y}{x^2 + y^2 + 1}, \quad x_3 = 1 - \frac{2}{x^2 + y^2 + 1}.$$

We can then simplify this to

$$x_1 = \frac{2x}{x^2 + y^2 + 1}, \quad x_2 = \frac{2y}{x^2 + y^2 + 1}, \quad x_3 = \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}, \tag{1.11}$$

which gives an explicit formula for stereographic projection

$$\Phi : \mathbb{C} \longrightarrow S^2 - \{N\}, \quad \Phi(z) = \left(\frac{2\mathrm{Re}(z)}{|z|^2 + 1}, \frac{2\mathrm{Im}(z)}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1}\right),$$

where $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ are the real and imaginary parts of z.

To see that Φ is one-to-one and onto, we can construct an explicit inverse. Indeed, eliminating t from equations (1.10) gives first $t = 1 - x_3$ and then

$$x = \frac{x_1}{t} = \frac{x_1}{1 - x_3}, \quad y = \frac{x_2}{t} = \frac{x_2}{1 - x_3}.$$

Thus we can define a function

$$\phi : S^2 - \{N\} \to \mathbb{C} \quad \text{by} \quad \phi(x_1, x_2, x_3) = \frac{x_1 + ix_2}{1 - x_3}.$$

which is exactly the inverse of Φ. The inverse map ϕ is well-behaved except when $x_3 = 1$, that is, it is well-behaved except at the north pole N on S^2.

These explicit formulae may seem confusing at first. The important point to note is that as the modulus of z gets larger and larger, $\Phi(z)$ approaches the north pole on S^2. Thus if we think of using Φ to identify points of $\mathbb{C}$ with points on $S^2 - \{N\}$, then the north pole N should be identified with a point at infinity. Indeed we might think of extending Φ to a map

$$\tilde{\Phi} : \mathbb{C} \cup \{\infty\} \longrightarrow S^2.$$

This idea of adding a point at infinity to the complex plane, thereby obtaining what is sometimes called *the extended complex plane* or the *one-point compactification* of $\mathbb{C}$, has turned out to be extremely useful in understanding functions of a complex variable.

Example. Supppose that $U = \mathbb{C} - \{3\}$ and

$$f : U \to \mathbb{C} \quad \text{by} \quad f(z) = \frac{1}{z - 3}.$$

Since $f(z)$ gets larger and larger as z approaches 3, it is often useful to extend f to a map

$$\hat{f} : \mathbb{C} \to \mathbb{C} \cup \{\infty\} \quad \text{so that} \quad f(z) = \begin{cases} 1/(z-3), & \text{if } z \neq 3, \\ \infty, & \text{if } z = 3. \end{cases}$$

Unfortunately, there is no way of defining addition and multiplication with ∞ so that $\mathbb{C} \cup \{\infty\}$ satisfies the field axioms.

We can think of ϕ as defining a coordinate z on S^2, which is well-behaved everywhere except at the north pole N. But sometimes we want a coordinate w that might be well-behaved near ∞. We might try to take

$$w = \frac{1}{z}$$

as such a coordinate. Can we think of this also as a complex coordinate on part of S^2? We find that

$$\begin{aligned}
z = \frac{x_1 + ix_2}{1 - x_3} \\
&\Rightarrow \quad w = \frac{1}{z} = \frac{1 - x_3}{x_1 + ix_2} = \frac{(1 - x_3)(x_1 - ix_2)}{x_1^2 + x_2^2} \\
&\qquad\qquad = \frac{(1 - x_3)(x_1 - ix_2)}{1 - x_3^2} = \frac{x_1 - ix_2}{1 + x_3}
\end{aligned}$$

Thus if we let S denote the south pole $(0, 0, -1)$, we can define

$$\psi : S^2 - \{S\} \to \mathbb{C} \quad \text{by} \quad w = \psi(x_1, x_2, x_3) = \frac{x_1 - ix_2}{1 + x_3}.$$

It tuns out that conjugation followed by the inverse of ψ is then stereographic projection with the north pole replaced by the south pole.

These constructions give two complex coordinate systems on S^2,

$$\phi : S^2 - \{N\} \to \mathbb{C} \quad \text{and} \quad \psi : S^2 - \{S\} \to \mathbb{C}$$

which are related by

$$\psi \circ \phi^{-1}(z) = w = \frac{1}{z}.$$

These complex coordinate systems make S^2 into what is called a *Riemann surface*.

Chapter 2

Analytic functions

Recall that if A and B are sets, a function $f : A \to B$ is a rule which assigns to each element $a \in A$ a unique element $f(a) \in B$. In this course, we will usually be concerned with complex-valued functions of a complex variable, functions $f : U \to \mathbb{C}$, where U is an open subset of $\mathbb{C}$. For such a function, we will often write

$$w = f(z) = f(x + iy) = u(x, y) + iv(x, y),$$

where $u(x, y)$ and $v(x, y)$ are the real and imaginary parts of f. We can think of the complex-valued function f as specified by these two real-valued functions u and v.

For example, if $f : \mathbb{C} \to \mathbb{C}$ is defined by $w = f(z) = z^2$, then

$$z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy \quad \Rightarrow \quad \begin{cases} u(x, y) = x^2 - y^2, \\ v(x, y) = 2xy. \end{cases}$$

We can think of this as defining a transformation

$$u = x^2 - y^2, \quad v = 2xy$$

from the (x, y)-plane to the (u, v)-plane.

Our goal is to study complex analytic functions $f : U \to \mathbb{C}$, functions which have a complex derivative at each point of U. We will see that the existence of a complex derivative at every point is far more restrictive than the existence of derivatives of real valued functions. We will also see that a function $f : U \to \mathbb{C}$ is complex analytic if and only if its component functions u and v have continuous partial derivatives and satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Moreover, $u(x, y)$ and $v(x, y)$ automatically have arbitrarily many derivatives and satisfy Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

These equations have many practical applications. For example, the real part of an analytic function can be used to model steady-state temperature distributions in regions of the plane.

2.1 Convergence and continuity

To properly deal with complex-valued functions, we need to understand limits and continuity. These are similar to the same concepts for real-valued functions which are studied informally in calculus or more carefully in real analysis courses such as Math 117 (see [8]). The simplest of the definitions is that of limit of a complex sequence.

Definition 1. A sequence (z_n) of complex numbers is said to *converge* to a complex number z if for every $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$n \in \mathbb{N} \quad \text{and} \quad n > N \quad \Rightarrow \quad |z_n - z| < \varepsilon.$$

In this case, we write $z = \lim z_n$. A sequence (z_n) of real numbers which does not converge to a real number is said to *diverge*.

Example 1. We claim that the sequence (z_n) defined by $z_n = 1/n$ converges to 0. Indeed, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $N > 1/\varepsilon$ and thus $1/N < \varepsilon$ by the so-called Archimedean property of the real numbers. It follows that

$$n > N \quad \Rightarrow \quad 0 < \frac{1}{n} < \frac{1}{N} \quad \Rightarrow \quad |z_n - 0| = \left| \frac{1}{n} - 0 \right| < \varepsilon.$$

Using the same technique, you could show that the sequence (z_n) defined by $z_n = c/n$ converges to 0, whenever c is a complex number.

Example 2. On the other hand, the sequence (z_n) defined by $z_n = i^n$ diverges. We can prove this by contradiction. Suppose that this sequence (z_n) were to converge to z. We could then take $\varepsilon = 1$, and there would exist $N \in \mathbb{N}$ such that

$$n > N \quad \Rightarrow \quad |z_n - z| < 1.$$

But then if $n > N$ and n is even, we would have $z_{n+2} = (i^2)z_n = -z_n$. Since $|z_n| = 1$,

$$2 = |z_n - z_{n+2}| \leq |z_n - z| + |z - z_{n+2}| < 1 + 1 = 2,$$

a contradiction.

Example 3. Suppose that $z_n = a^n$, where $a \in \mathbb{R}$ and $0 < a < 1$. Then

$$\frac{1}{a} = 1 + b, \quad \text{where} \quad b > 0,$$

and by the binomial formula

$$\left(\frac{1}{a} \right)^n = (1+b)^n = 1 + nb + \cdots + b^n \geq nb,$$

so

$$a^n \leq \frac{1}{bn} = \frac{c}{n}, \quad \text{where} \quad c = \frac{1}{b}.$$

Now the argument for Example 1 can be applied with the result that

$$\lim z_n = \lim(a^n) = 0.$$

Finally if $c \in \mathbb{C}$ and $|c| < 1$, then

$$|c^n - 0| \leq |c|^n \quad \Rightarrow \quad \lim c^n = 0.$$

Proposition 1. *Suppose that (z_n) and (w_n) are convergent sequences of complex numbers with $\lim z_n = z$ and $\lim w_n = w$. Then*

1. *$(z_n + w_n)$ converges and $\lim(z_n + w_n) = z + w$,*
2. *$(z_n w_n)$ converges and $\lim(z_n w_n) = zw$,*
3. *(z_n/w_n) converges and $\lim(z_n/w_n) = z/w$, provided $w_n \neq 0$ for all n and $w \neq 0$.*

This theorem is proven in Math 117 for sequences of real numbers, and exactly the same proof holds for sequences of complex numbers. For example, to prove part I, we let $\varepsilon > 0$ be given. Since (z_n) converges to z, there exists an $N_1 \in \mathbb{N}$ such that

$$n \in \mathbb{N} \quad \text{and} \quad n > N_1 \quad \Rightarrow \quad |z_n - z| < \frac{\varepsilon}{2}.$$

Since (w_n) converges to w, there exists an $N_2 \in \mathbb{N}$ such that

$$n \in \mathbb{N} \quad \text{and} \quad n > N_2 \quad \Rightarrow \quad |w_n - w| < \frac{\varepsilon}{2}.$$

Let $N = \max(N_1, N_2)$. Then using the triangle inequality, we conclude that

$$n \in \mathbb{N} \quad \text{and} \quad n > N \quad \Rightarrow \quad |(z_n + w_n) - (z + w)| \leq |z_n - z| + |w_n - w| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which is exactly what we needed to prove. This is sometimes called the $\varepsilon/2$ trick. Similar arguments are used to prove the other parts of the Proposition.

Example 4. Suppose we want to investigate the convergence of the sequence (z_n) defined by

$$z_n = \frac{2n + 3 + i}{n + 5 - 4i}.$$

We can rewrite this as

$$z_n = \frac{2 + (3 + i)/n}{1 + (5 - 4i)/n}.$$

By Example 1, we see that as $n \to \infty$, $(3+i)/n$ and $(5-4i)/n$ converge to zero. We can then use the above Proposition to establish that

$$\lim(2 + (3 + i)/n) = 2, \quad \lim(1 + (5 - 4i)/n) = 1 \quad \text{and} \quad \lim z_n = 2.$$

Similarly, we can use the Proposition to show that the sequence (z_n) defined by

$$z_n = \frac{2n^2 + (3+5i)n + (6-i)}{3n^2 + (5+i)n + (2+3i)}$$

converges to $z = 2/3$.

Example 5. Suppose that (z_n) is the sequence of complex numbers defined by

$$z_1 = 1.5, \quad z_{n+1} = f(z_n) = \frac{z_n}{2} + \frac{1}{z_n}, \quad \text{for } n \in \mathbb{N}. \tag{2.1}$$

If this sequence has a limit, Proposition 1 tells us what the limit must be. Indeed, by induction one sees that $z_n \in \mathbb{R}$ and $z_n > 0$ for all $n \in \mathbb{N}$. Moreover, using calculus, one can show that $x \in \mathbb{R} \;\Rightarrow\; f(x) \geq \sqrt{2}$. Thus if $z = \lim z_n$, it follows from Proposition 1 that

$$z = \lim z_{n+1} = \frac{\lim z_n}{2} + \frac{1}{\lim z_n} = \frac{z}{2} + \frac{1}{z} \quad \Rightarrow \quad \frac{z}{2} = \frac{1}{z} \quad \Rightarrow \quad z^2 = 2.$$

Thus we see that $z = \sqrt{2}$. We remark that (2.1) provides a good numerical method for finding the square root of two.

We next turn to the notion of limits of functions. If z_0 is a complex number, a *deleted open ball* about z_0 is a set of the form

$$N^*(z_0; \varepsilon) = N(z_0; \varepsilon) - \{z_0\} = \{z \in \mathbb{C} : 0 < |z - z_0| < \varepsilon\},$$

for some $\varepsilon > 0$.

Definition 2. Let $D \subseteq \mathbb{C}$ and let z_0 be a complex number such that some deleted neighborhood $N^*(z_0; \varepsilon)$ of z_0 lies within D. A complex number w_0 is the *limit* of a function $f : D \to \mathbb{C}$ at z_0 if

$$z_n \in D, \quad z_n \neq z_0 \quad \text{and} \quad \lim z_n = z \quad \Rightarrow \quad \lim f(z_n) = w_0.$$

In this case, we write

$$\lim_{z \to z_0} f(z) = w_0.$$

Some authors, including [10], prefer an alternate definition which turns out to be equivalent:

Definition 2'. Let $D \subseteq \mathbb{C}$ and let z_0 be a complex number such that some deleted neighborhood of z_0 lies within D. A complex number w_0 is the *limit* of a function $f : D \to \mathbb{C}$ at z_0 if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < |z - z_0| < \delta \quad \Rightarrow \quad |f(z) - w_0| < \varepsilon.$$

With either Definition 2 or 2', it is important that the function f need be defined only on a deleted open ball about z_0, not necessarily at the point z_0

itself. Indeed, we will often take limits at z_0 when f is not defined at z_0. The following proposition follows immediately from Definition 2 and Proposition 1:

Proposition 2. *Let $D \subseteq \mathbb{C}$ and let z_0 be a complex number such that some deleted neighborhood of z_0 lies within D. Suppose that $f : D \to \mathbb{C}$ and $g : D \to \mathbb{C}$ are functions such that $\lim_{z\to z_0} f(z) = w_0$ and $\lim_{z\to z_0} g(z) = w_1$. Then*

1. $\lim_{z\to z_0}(f(z) + g(z)) = w_0 + w_1$,
2. $\lim_{z\to z_0}(f(z)g(z)) = w_0 w_1$,
3. *if $g(z) \neq 0$ for $z \in D$ and $w_1 \neq 0$, then* $\lim_{z\to z_0}(f(z)/g(z)) = w_0/w_1$.

Definition 3. Suppose that $D \subseteq \mathbb{C}$, that $f : D \to \mathbb{C}$, and z_0 is a point of D such that $N(z_0; \varepsilon) \subseteq D$ for some $\varepsilon > 0$. Then f is continuous at z_0 if

$$\lim_{z\to z_0} f(z) = f(z_0).$$

The following proposition follows immediately from this definition and Proposition 2:

Proposition 3. *Suppose that $D \subseteq \mathbb{C}$ and that $f : D \to \mathbb{C}$ and $g : D \to \mathbb{C}$ are continuous at $z_0 \in D$. Then the functions $f + g$ and $f \cdot g$ are also continuous at z_0. Moreover, if $g(z) \neq 0$ for $z \in D$, then the quotient f/g is also continuous at z_0.*

It is quite easy to show that the function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = z$ is continuous at every $z_0 \in \mathbb{C}$. It then follows from Proposition 3 that every polynomial function

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

with complex coefficients a_0, a_1, $\ldots$, a_{n-1}, a_n, is continuous at every $z_0 \in \mathbb{C}$. Suppose that

$$Q(z) = b_m z^m + b_{m-1} z^{m-1} + \cdots + b_1 z + b_0 = 0$$

is a second polynomial with complex coefficients and that S be the finite set of points at which $Q(z)$ vanishes. Let $D = \mathbb{C} - S$. Then it also follows from Proposition 3 that the *rational function* $R : D \to \mathbb{C}$ defined by

$$R(z) = \frac{P(z)}{Q(z)}$$

is continuous at every point of D. Thus we can construct many examples of continuous functions. Moreover, it is easy to calculate the limits of continuous functions, because if f is continuous at z_0 then

$$\lim_{z\to z_0} f(z) = f(z_0).$$

But there are also numerous cases in which limits do not exist. For an important example, suppose that $D = \mathbb{C} - \{0\}$ and that $f : D \to \mathbb{C}$ is defined by

$$f(x + iy) = f(z) = \frac{\bar{z}}{z} = \frac{x - iy}{x + iy} = \frac{(x - iy)^2}{x^2 + y^2} = \frac{x^2 - y^2 - 2xyi}{x^2 + y^2}.$$

Then if $(z_n = x_n + iy_n)$ is a sequence in D which lies on the x-axis, $y_n = 0 \Rightarrow f(z_n) = 1$, while if $(z_n = x_n + iy_n)$ is a sequence in D which lies on the y-axis, $x_n = 0 \Rightarrow f(z_n) = -1$. Thus $\lim_{z \to 0} f(z)$ does not exist.

Here is another example. Recall that $\mathrm{Arg}(z)$ is the unique value of the multivalued angle function $\arg(z)$ which lies in the interval $(-\pi, \pi]$. This defines a function

$$\mathrm{Arg} : \mathbb{C} - \{0\} \to (-\pi, \pi]$$

which we use to define the logarithm

$$\mathrm{Log} : \mathbb{C} - \{0\} \to \mathbb{C} \quad \text{by} \quad \mathrm{Log}(z) = \mathrm{Log}|z| + i\mathrm{Arg}(z). \tag{2.2}$$

As we saw earlier, if $\exp : \mathbb{C} \to \mathbb{C}$ is the function defined by $\exp(z) = e^z$, then

$$\exp \circ \mathrm{Log}(z) = z, \qquad \mathrm{Log} \circ \exp(w) = w,$$

when the composition is defined. If $(z_n = x_n + iy_n)$ is a sequence of complex numbers such that $y_n > 0$ and $\lim z_n = -1$, then $\lim \mathrm{Log}(z_n) = \pi$, while if $(z_n = x_n + iy_n)$ is a sequence such that $y_n < 0$ and $\lim z_n = -1$, then $\lim \mathrm{Log}(z_n) = -\pi$. Thus

$$\lim_{z \to -1} \mathrm{Log}(z)$$

does not exist, and the function Log defined by (2.2) fails to be continuous at $z = -1$.

On the other hand, one can show that the restricted function

$$\mathrm{Log} : \mathbb{C} - \{x \in \mathbb{R} : x \leq 0\} \to \mathbb{C}, \quad \text{defined by} \quad \mathrm{Log}(z) = \mathrm{Log}|z| + i\mathrm{Arg}(z),$$

is in fact continuous, because we have excised the points of discontinuity from the domain.

2.2 Complex derivatives and analyticity

The notion of derivative for a complex-valued function looks superficially similar to the similar definition for real-valued functions, but we will see that it has far stronger implications:

Definition 1. Suppose that $f : U \to \mathbb{C}$ is a complex valued function, where U is an open subset of $\mathbb{C}$ and $z_0 \in U$. Then the *complex derivative* of f at z_0 is

$$\frac{df}{dz}(z_0) = f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}, \tag{2.3}$$

if this limit exists. We say that f is *differentiable* at z_0 if it has a complex derivative at z_0.

Definition 2. A function $f : U \to \mathbb{C}$, where U is an open subset of $\mathbb{C}$, is said to be *complex analytic* or *holomorphic* if it has a derivative at every $z_0 \in U$. If $U = \mathbb{C}$, we say that the function is *entire*.

Example 1. If $f : \mathbb{C} \to \mathbb{C}$ is the function defined by $f(z) = z^n$, then it follows from the binomial formula that

$$f(z_0 + \Delta z) = z_0^n + n z_0^{n-1} \Delta z + \binom{n}{2} z_0^{n-2} (\Delta z)^2 + \cdots + (\Delta z)^n,$$

so

$$f(z_0 + \Delta z) - f(z_0) = n z_0^{n-1} \Delta z + \binom{n}{2} z_0^{n-2} (\Delta z)^2 + \cdots + (\Delta z)^n$$

and

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = n z_0^{n-1} + \binom{n}{2} z_0^{n-2} (\Delta z) + \cdots + (\Delta z)^{n-1}.$$

Taking the limit as $\Delta z \to 0$ yields

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = n z_0^{n-1}.$$

Thus in this case, the derivative of f exists at every $z_0 \in \mathbb{C}$ and is given by the familiar formula

$$f'(z_0) = n z_0^{n-1}.$$

Thus f is complex analytic on the entire complex plane, that is, it is an entire function.

Example 2. If $f : \mathbb{C} \to \mathbb{C}$ is the function defined by $f(z) = \bar{z}$, then

$$f(z_0 + \Delta z) - f(z_0) = \overline{(z_0 + \Delta z)} - \bar{z}_0 = \overline{\Delta z}.$$

Thus

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z}.$$

But

$$\Delta z \in \mathbb{R} \quad \Rightarrow \quad \frac{\overline{\Delta z}}{\Delta z} = 1 \quad \text{while} \quad \Delta z \in i\mathbb{R} \quad \Rightarrow \quad \frac{\overline{\Delta z}}{\Delta z} = -1,$$

so

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

cannot exist in this case, so f is not differentiable at any $z_0 \in \mathbb{C}$, even though if we divide f into real and imaginary parts,

$$f(z) = f(x + iy) = u(x, y) + iv(x, y), \quad \text{then} \quad \begin{cases} u(x, y) = x, \\ v(x, y) = -y \end{cases}$$

and the component functions u and v have continuous partial derivatives of arbitrarily high order in the usual sense of several variable calculus.

One can use Proposition 2 from §2.1 as the foundation for proving:

Proposition 1. *If $f : D \to \mathbb{C}$ and $g : D \to \mathbb{C}$ are differentiable at $z_0 \in D$, then*

1. *$f + g$ is differentiable at z_0, and $(f + g)'(z_0) = f'(z_0) + g'(z_0)$,*
2. *cf is differentiable at z_0 for any constant c, and $(cf)'(z_0) = cf'(z_0)$,*
3. *fg is differentiable at z_0, and $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$,*
4. *if $g(z) \neq 0$ for $z \in D$, then f/g is differentiable at z_0. and*

$$\left(\frac{f}{g}\right)'(z_0) = \frac{g(z_0)f'(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}. \tag{2.4}$$

Using Example 1 and this Proposition, it becomes straightforward to show that any polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

with complex coefficients a_0, a_1, ..., a_{n-1}, a_n is differentiable at every point $z_0 \in \mathbb{C}$, with complex derivative given by the formula

$$P'(z_0) = n a_n z_0^{n-1} + n(n-1) a_{n-1} z_0^{n-2} + \cdots + a_1.$$

Moreover, if

$$Q(z) = b_m z^m + b_{m-1} z^{m-1} + \cdots + b_1 z + b_0 = 0$$

is a second polynomial with complex coefficients and S is the finite set of points at which $Q(z)$ vanishes. It also follows from Proposition 1 that the rational function

$$R(z) = \frac{P(z)}{Q(z)}$$

is differentiable at every point of $D = \mathbb{C} - S$. In this case, we would say that R is complex analytic except at the points of S, and we can use (2.4) to find the derivative at any point of $\mathbb{C} - S$.

We can also prove a version of the chain rule for complex derivatives:

Proposition 2. *Suppose that U and V are open subsets of the complex plane $\mathbb{C}$ and that $f : U \to V$ and $g : V \to \mathbb{C}$ are differentiable at $z_0 \in U$ and $f(z_0) \in V$ respectively. Then the composition $g \circ f : U \to \mathbb{C}$ is differentiable at z_0 and*

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0).$$

Here is a simple application of the chain rule: We will see in the next section that the function

$$g : \mathbb{C} \to \mathbb{C} \quad \text{defined by} \quad g(x+iy) = e^z = e^x(\cos y + i \sin y)$$

is differentiable at every point of $\mathbb{C}$ and $g'(z) = e^z$. If $f : \mathbb{C} \to \mathbb{C}$ is defined by $f(w) = aw$, where $a \in \mathbb{C}$, then $g \circ f(z) = e^{az}$, and it follows from the chain rule that

$$(g \circ f)'(z) = g'(f(z))f'(z) = e^{az}a = ae^{az}.$$

Finally, just as in the real case, it turns out that a function which has a complex derivative at a point automatically is continuous at that point:

Proposition 3. *If $f : D \to \mathbb{C}$ is differentiable at $z_0 \in D$, then f is continuous at z_0 as well.*

To prove this, we let $z = z_0 + \Delta z$ in (2.3) so that $\Delta z = z - z_0$. Then

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

Thus

$$\begin{aligned} \lim_{z \to z_0} (f(z) - f(z_0)) &= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \lim_{z \to z_0} (z - z_0) \\ &= f'(z_0) \lim_{z \to z_0} (z - z_0) = 0. \end{aligned}$$

But this immediately implies that f is continuous at z_0.

Thus if D is an open subset of $\mathbb{C}$ and $f : D \to \mathbb{C}$ is complex analytic, then f is continuous. It is a little harder to show (via a theorem of Goursat) that a complex analytic function $f : D \to \mathbb{C}$ automatically has a continuous derivative $f' : D \to \mathbb{C}$. Later we will see via the Cauchy integral theorem that in fact a complex analytic function has continuous derivatives of arbitrarily high order. Thus existence of complex derivatives is far stronger than the existence of ordinary derivatives of real valued functions.

2.3 The Cauchy-Riemann equations

Suppose that the function $f : D \to \mathbb{C}$ has a complex derivative at $z_0 = x_0 + iy_0 \in D$. In the definition of complex derivative, we can let Δz approach zero along the x-axis, that is, we can set $\Delta z = h \in \mathbb{R}$. In that case, (2.3) becomes

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h},$$

where h ranges only over $\mathbb{R}$. We recognize that this is just the partial derivative of the vector valued function $f(x+iy) = u(x,y) + iv(x,y)$ with respect to x, which we can write out in terms of real and imaginary parts:

$$f'(z_0) = \frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0). \tag{2.5}$$

On the other hand, we can also let Δz approach zero along the y-axis, that is, we can set $\Delta z = ik$, where $k \in \mathbb{R}$. In this case, we find that

$$\begin{aligned} f'(z_0) &= \lim_{k\to 0} \frac{f(z_0+ik)-f(z_0)}{ik} = \frac{1}{i}\frac{\partial f}{\partial y}(x_0,y_0) \\ &= \frac{1}{i}\frac{\partial u}{\partial y}(x_0,y_0) + \frac{\partial v}{\partial y}(x_0,y_0) = -i\frac{\partial u}{\partial y}(x_0,y_0) + \frac{\partial v}{\partial y}(x_0,y_0). \end{aligned} \tag{2.6}$$

Since the two expressions (2.5) and (2.6) must be equal, we must have

$$\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0) \quad \text{and} \quad \frac{\partial u}{\partial y}(x_0,y_0) = -\frac{\partial v}{\partial x}(x_0,y_0).$$

Thus the real and imaginary parts of a complex analytic function must satisfy these equations, known as the *Cauchy-Riemann equations* in honor of Augustin Cauchy (1789-1859) and Bernard Riemann (1826-1866), although these equations had actually appeared earlier in work of d'Alembert and Euler on fluid motion (as explained in Chapter 26 of [5]).

Conversely, we have the following key theorem:

Cauchy-Riemann Theorem. *Suppose that U is an open subset of $\mathbb{C}$ and the complex-valued function $f : U \to \mathbb{C}$ can be expressed in terms of real and imaginary parts as*

$$f(z) = f(x+iy) = u(x,y) + iv(x,y),$$

where $u(x,y)$ and $v(x,y)$ have continuous first order partial derivatives on U which satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{2.7}$$

Then f is a complex analytic function on U and its derivative is given by the formula

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}. \tag{2.8}$$

We sketch a proof of this following Ahlfors [1], page 26. It is proven in calculus courses (or more rigorously in real analysis courses) that when $u(x,y)$ and $v(x,y)$ have continuous partial derivatives,

$$u(x+h,y+k) - u(x,y) = \frac{\partial u}{\partial x}(x,y)h + \frac{\partial u}{\partial y}(x,y)k + \varepsilon_1$$

and

$$v(x+h,y+k) - v(x,y) = \frac{\partial v}{\partial x}(x,y)h + \frac{\partial v}{\partial y}(x,y)k + \varepsilon_2,$$

where

$$\frac{\varepsilon_1}{\sqrt{h^2+k^2}} \to 0 \quad \text{and} \quad \frac{\varepsilon_2}{\sqrt{h^2+k^2}} \to 0 \quad \text{as} \quad \sqrt{h^2+k^2} \to 0.$$

Using (2.7), we can rewrite the above equations as

$$\begin{pmatrix} u(x+h,y+k) \\ v(x+h,y+k) \end{pmatrix} - \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v \partial x & \partial v/\partial y \end{pmatrix} (x,y) \begin{pmatrix} h \\ k \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}. \quad (2.9)$$

The key point now is that the Jacobian matrix

$$\begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v \partial x & \partial v/\partial y \end{pmatrix} (x,y) \quad \text{represents a complex matrix} \quad \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

if and only if the Cauchy-Riemann equations are satisfied. Thus if the Cauchy-Riemann equations are satisfied, we can rewrite (2.9) as

$$\begin{pmatrix} u(x+h,y+k) \\ v(x+h,y+k) \end{pmatrix} - \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ -\partial u/\partial y & \partial u/\partial x \end{pmatrix} (x,y) \begin{pmatrix} h \\ k \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}.$$

One can check that the two components of this last equation are the real and imaginary parts of the complex equation

$$f(z + (h+ik)) - f(z) = \left(\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right) (h+ik) + \varepsilon_1 + i\varepsilon_2.$$

But this implies that

$$\lim_{h+ik \to 0} \left(\frac{f(z+(h+ik)) - f(z)}{h+ik} \right) = \left(\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right) (x,y),$$

so f is analytic and its derivative is given by (2.8).

Application 1. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is the exponential function defined by

$$f(x+iy) = e^z = e^x(\cos y + i \sin y) = u(x,y) + iv(x,y),$$

where $u(x,y) = e^x \cos y$ and $v(x,y) = e^x \sin y$. We could try to prove directly that this function is complex analytic, but this would require evaluating some difficult limits. Instead, we can observe the real and imaginary parts of f are continuously differentiable, and

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -e^x \sin y = \frac{\partial v}{\partial x},$$

so the Cauchy-Riemann Theorem implies that f is a complex analytic function. Moreover,

$$\frac{df}{dz} = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} = e^x \cos y + i e^x \sin y, \quad \text{so} \quad \frac{d}{dz}(e^z) = e^z.$$

Exercise B. a. Use the chain rule to express the partial derivatives of u and v with respect to x and y in terms of the partial derivatives with respect to the polar coordinates (r, θ), where

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

b. Use the expressions you obtained to rewrite the Cauchy-Riemann equations in terms of polar coordinates:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}. \tag{2.10}$$

Application 2. Suppose that

$$f = \mathrm{Log} : \mathbb{C} - \{x \in \mathbb{R} : x \leq 0\} \to \mathbb{C}, \quad \text{is defined by} \quad \mathrm{Log}(z) = \mathrm{Log}|z| + i\mathrm{Arg}(z),$$

In this case,

$$u(x, y) = \mathrm{Log}\left(\sqrt{x^2 + y^2}\right), \quad v(x, y) = \arg(x + iy).$$

In terms of polar coordinates,

$$u(r, \theta) = \mathrm{Log}(r), \quad v(r, \theta) = \theta,$$

so

$$\frac{\partial u}{\partial r} = \frac{1}{r}, \quad \frac{\partial u}{\partial \theta} = 0, \quad \frac{\partial v}{\partial r} = 0, \quad \frac{\partial u}{\partial \theta} = 1.$$

It therefore follows from the polar coordinate form of the Cauchy-Riemann equations (2.10), together with the Theorem, that $f = \mathrm{Log}$ is a complex analytic function of z, and using (2.8) we find its derivative:

$$\begin{aligned} f'(z) &= \frac{\partial}{\partial x}\mathrm{Log}\left(\sqrt{x^2 + y^2}\right) - i\frac{\partial}{\partial y}\mathrm{Log}\left(\sqrt{x^2 + y^2}\right) \\ &= \frac{1}{\sqrt{x^2 + y^2}}\frac{\partial}{\partial x}\left(\sqrt{x^2 + y^2}\right) - \frac{i}{\sqrt{x^2 + y^2}}\frac{\partial}{\partial y}\left(\sqrt{x^2 + y^2}\right) \\ &= \frac{x - iy}{x^2 + y^2} = \frac{\bar{z}}{\bar{z}z} = \frac{1}{z}. \end{aligned}$$

The following Proposition follows from Proposition 2 in §1.5:

Proposition 1. *Suppose that D is a connected open subset of $\mathbb{C}$ and that $f : D \to \mathbb{C}$ is a complex analytic function such that $f' : D \to \mathbb{C}$ is continuous. Then*

$$f'(z) = 0 \quad \text{for all } z \in D \quad \Rightarrow \quad f(z) \equiv c,$$

where c is a constant.

Indeed, if $f'(z) = 0$ for all $z \in D$, then it follows from (2.8) and the Cauchy-Riemann equations that

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0, \qquad \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0.$$

Since D is connected, it follows from Proposition 2 in §1.5, or from Theorem 1, page 40 in [10], that u and v are both constant.

Many properties of complex analytic functions can be derived from the Cauchy-Riemann equations. For example:

Proposition 2. *Suppose that D is a connected open subset of $|mathbbC$ and that $f : U \to \mathbb{C}$ is a complex analytic function such that $f' : D \to \mathbb{C}$ is continuous. If the real part of f is constant, then f itself is constant.*

Proof: Suppose that $f(x+iy) = u(x,y) + iv(x,y)$, so that u is the real part of f. Then

$$u \text{ constant} \quad \Rightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0.$$

It then follows from the Cauchy-Riemann equations that

$$\frac{\partial v}{\partial x} = -\frac{\partial v}{\partial y} = 0, \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} = 0.$$

It therefore follows from Proposition 2 in §1.5 that v is constant. Thus f itself is constant. QED

Can you use this proposition to show that if $f : D \to \mathbb{C}$ and $g : D \to \mathbb{C}$ are two complex analytic functions with the same real part, then $f = g + c$, where c is an imaginary constant?

2.4 Fluid motion in the plane

In his research on complex analysis, Riemann utilized physical models to buttress his intuition, as emphasized by Felix Klein in his classic treatise [4] on Riemann's theory of complex functions. One of the models Riemann used was that of a fluid flow tangent to the field lines for an electric field in the (x,y)-plane, a flow which turns out to be both incompressible and irrotational.

One can represent the velocity of a fluid in an open subset U of the (x,y)-plane by a vector field

$$\mathbf{V}(x,y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j} : U \longrightarrow \mathbb{R},$$

where $\mathbf{i}$ and $\mathbf{j}$ are the perpendicular unit-length vectors pointing in the x and y coordinate directions. We say that $\mathbf{V}$ has continuous first-order partial derivatives if its component functions M and N have continuous first-order partial derivatives. Such a vector field $\mathbb{V}$ (or the corresponding fluid) is said to be *incompressible* if

$$\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 0, \tag{2.11}$$

and *irrotational* if

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0. \tag{2.12}$$

An important fact is that we can rotate the vector field $\mathbf{V}$ counterclockwise through 90 degrees, obtaining

$$\star\mathbf{V} = -N(x,y)\mathbf{i} + M(x,y)\mathbf{j}, \tag{2.13}$$

and one can check that

$$\mathbf{V} \text{ is irrotational} \quad \Leftrightarrow \quad \star\mathbf{V} \text{ is incompressible},$$

$$\mathbf{V} \text{ is incompressible} \quad \Leftrightarrow \quad \star\mathbf{V} \text{ is irrotational}.$$

We will see later that the velocity field for a steady-state fluid of constant density is incompressible so long as no fluid is being created or destroyed. On the other hand, if U is either $\mathbb{C}$ or an open ball in $\mathbb{C}$ then a vector field is irrotational if and only if it is the gradient of a function:

Poincaré Lemma. *Suppose that $U = \mathbb{C}$ or*

$$U = N((x_0, y_0); R) = \{x + iy \in \mathbb{C} : d((x, y), (x_0, y_0)) < R\},$$

for some $(x_0, y_0) \in \mathbb{C}$ and some $R > 0$, and that $\mathbf{V}$ is is a vector field on U with continuous first-order partial derivatives. Then

$$\mathbf{V} = M\mathbf{i} + N\mathbf{j} = \nabla u, \quad \textit{for some } u : U \to \mathbb{R} \quad \Leftrightarrow \quad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0.$$

Here the condition $\mathbf{V} = \nabla u$ means that

$$\frac{\partial u}{\partial x}(x, y) = M(x, y), \quad \frac{\partial u}{\partial y}(x, y) = N(x, y). \tag{2.14}$$

One direction of the proof is easy. If $\mathbf{V} = \nabla u$, then (2.14) and equality of mixed partials yields

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = 0.$$

We will not prove the other direction of the Poincaré Lemma now except to note that it follows from Green's Theorem from vector calculus. Instead, we note that one constructs the function $u : U \to \mathbb{R}$ such that $\mathbf{V} = \nabla u$ by the *method of exact differentials*. Indeed, with sufficient effort, one could make the method of exact differentials into a proof of the Poincaré Lemma.

To express the Poincaré Lemma in the language of differentials, we let

$$d\mathbf{x} = dx\mathbf{i} + dy\mathbf{j}, \quad \text{so that} \quad \nabla u \cdot d\mathbf{x} = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = du$$
$$\text{and} \quad \mathbf{V}(x, y) \cdot d\mathbf{x} = (M\mathbf{i} + N\mathbf{j}) \cdot (dx\mathbf{i} + dy\mathbf{j}) = Mdx + Ndy,$$

and hence

$$\mathbf{V} = \nabla u \quad \Leftrightarrow \quad Mdx + Ndy = du,$$

and the Poincaré Lemma becomes

$$Mdx + Ndy = du \quad \Leftrightarrow \quad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0.$$

Example 1. Suppose that

$$\mathbf{V}(x,y) = (4x^3 - 12xy^2)\mathbf{i} + (-12x^2y + 4y^3)\mathbf{j}$$
$$\text{so that} \quad M dx + N dy = (4x^3 - 12xy^2)dx + (-12x^2y + 4y^3)dy.$$

Then
$$\frac{\partial N}{\partial x} = -12x^2 + 12y^2 = \frac{\partial M}{\partial y},$$
so $\mathbf{V} = \nabla u$, for some function u by the Poincaré Lemma. To find u, first note that
$$\frac{\partial u}{\partial x} = 4x^3 - 12xy^2 \quad \Rightarrow \quad u(x,y) = x^4 - 6x^2y^2 + g(y),$$
where $g(y)$ is a function of y alone, while
$$\frac{\partial u}{\partial y} = -12x^2y + 4y^3 \quad \Rightarrow \quad -12x^2y + g'(y) = -12x^2y + 4y^3,$$
so
$$g(y) = y^4 + c \quad \text{and} \quad u(x,y) = x^4 - 6x^2y^2 + y^4 + c,$$
where c is a constant. Thus $\mathbf{V}$ determines u up to a constant. We call u a *potential* for the fluid flow $\mathbf{V}$.

Suppose now that $\mathbf{V} : U \to \mathbb{R}$ is both incompressible and irrotational. If U is the entire plane or an open ball, it follows from the Poincaré Lemma that $\mathbf{V}$ has a potential u and substituting (2.14) into (2.11) yields the *Laplace equation*:
$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{2.15}$$

Definition. A function $u : U \to \mathbb{R}$ with continuous first and second order partial derivatives is said to be *harmonic* if u satisfies (2.15).

Thus harmonic functions are exactly the potentials for irrotational incompressible fluid flows in the plane. What is important for complex analysis is that if
$$f(x+iy) = u(x,y) + iv(x,y)$$
is a complex analytic function with continuous partial derivatives up to order two on an open subset U of the complex plane $\mathbb{C}$, then u and v are harmonic functions. Not only that, but a harmonic function $u : U \to \mathbb{C}$ is the real part of a complex analytic function, at least if U is the entire complex plane or an open ball in the complex plane. Indeed, if u is harmonic, then its gradient
$$\mathbf{V} = \nabla u = \frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j}$$
is both irrotational and incompressible. But then
$$\star\mathbf{V} = -\frac{\partial u}{\partial y}\mathbf{i} + \frac{\partial u}{\partial x}\mathbf{j}$$

is also both irrotational and incompressible. If U is the entire complex plane or an open ball in the complex plane, we can apply the Poincaré Lemma and construct a potential $v : U \to \mathbb{R}$ for $\star\mathbf{V}$, so that

$$-\frac{\partial u}{\partial y}\mathbf{i} + \frac{\partial u}{\partial x}\mathbf{j} = \star\mathbf{V} = \frac{\partial v}{\partial x}\mathbf{i} + \frac{\partial v}{\partial y}\mathbf{j}.$$

But then u and v satisfy the Cauchy-Riemann equations, and (assuming the Poincar'e Lemma) we have proven:

Theorem. *Suppose that $U = \mathbb{C}$ or an open ball within U and that $u : U \to \mathbb{R}$ is a harmonic function on U. Then up to addition of a constant, there is a unique harmonic function $v(x, y)$ such that*

$$f(x + iy) = u(x, y) + iv(x, y), \quad \text{for } (x, y) \in U.$$

We call v the *harmonic conjugate* of u.

Example 2. Suppose that

$$u(x, y) = x^4 - 6x^2y^2 + y^4,$$

a function which is easily verified to be harmonic. Then

$$\mathbf{V} = \nabla u = (4x^3 - 12xy^2)\mathbf{i} + (-12x^2y + 4y^3)\mathbf{j},$$

so that

$$\star\mathbf{V} = (12x^2y - 4y^3)\mathbf{i} + (4x^3 - 12xy^2)\mathbf{j},$$

and we can find the harmonic conjugate v as follows:

$$\frac{\partial v}{\partial x} = 12x^2y - 4y^3 \quad \Rightarrow \quad v(x, y) = 4x^3y - 4xy^3 + g(y),$$

where $g(y)$ is a function of y alone. Then

$$\frac{\partial v}{\partial y} = 4x^3 - 12xy^2 \quad \Rightarrow \quad u(x, y) = 4x^3y - 4xy^3 + c,$$

where c is a constant. Thus if we set $c = 0$,

$$\begin{aligned} f(z) = f(x + iy) &= u(x, y) + iv(x, y) \\ &= x^4 - 6x^2y^2 + y^4 + i(4x^3y - 4xy^3) = (x + iy)^4 = z^4. \end{aligned}$$

We have seen that the study of irrotational incompressible fluid motion in the plane inexorably leads to the Cauchy-Riemann equations of complex analysis. Following Klein [4], we can imagine an electric field $\mathbf{V}(x, y)$ in the (x, y)-plane produced by a finite number of charges located at the points $\{z_1, \ldots, z_k\}$ (which will be singular points for $\mathbf{V}$). Maxwell's equations from electricity and magnetism imply that $\mathbf{V}$ is an incompressible irrotational flow on the open set

$$U = \mathbb{C} - \{z_1, \ldots, z_k\}.$$

However, even if we assume that $\mathbf{V}$ has a potential u, U is not an open ball within $\mathbb{C}$, so we cannot apply the Poincaré Lemma to construct a harmonic conjugate v. This raises the question: Can we extend the Poincaré Lemma to more general connected open sets U?

Exercise C. Using Exercise B, show that we can write Laplace's equation in polar coordinates as

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

Example 3. It follows from Exercise C, that the only solutions to Laplace's equation which are radially symmetric are the solutions to the ordinary differential equation

$$\frac{d^2 u}{dr^2} + \frac{1}{r}\frac{du}{dr} = 0.$$

If we set $w = du/dr$, this yields

$$\frac{dw}{dr} + \frac{w}{r} = 0 \quad \text{which has the solutions} \quad w = \frac{a}{r},$$

where a is a constant. This in turn implies that

$$u = a \text{ Log } r + b = a \text{ Log}|z| + b, \tag{2.16}$$

where a and b are constants; if $b = 0$ and $a > 0$, this is interpreted as a source at the origin $z = 0$ of strength a. In the electrostatic model, u is the potential produced by an electric charge placed at the origin.

But now we can ask the question: Does u have a harmonic conjugate? If we let

$$U = \mathbb{C} - \{x \in \mathbb{R} : x \leq 0\},$$

then the harmonic conjugate to u must be the imaginary part of the function Log we described before, which is given by

$$v(x, y) = \text{Arg}(x + iy),$$

while if $U = \mathbb{C} - \{0\}$, there is no continuous harmonic conjugate. (The harmonic conjugate would have to be the "multivalued function" arg, but that of course is not a genuine function.) We will return to the question of when the harmonic conjugate to a given harmonic function exists after we have studied contour integrals.

Remark: The stereographic projection $\Phi : \mathbb{C} \to S^2 - \{N\} \subseteq \mathbb{R}^3$ allows us to extend this model of fluid flow to the surface of the sphere S^2. Indeed, if $\phi : S^2 - \{N\} \to \mathbb{C}$ is the inverse to stereographic projection,

$$u \circ \phi : S^2 - \{N\} \longrightarrow \mathbb{C}$$

can be thought of as the potential for a fluid on $S^2 - \{N\}$, and the best behaved potentials are those which extend to the north pole N and are well-behaved near N.

For example, we could take the harmonic function $u = a \,\mathrm{Log}|z|$ of Example 3. How does this fluid flow behave near the north pole N? To answer that question, we write u in terms of the coordinate $w = 1/z$ which is well behaved near N:

$$u = a \ \mathrm{Log}|z| = a \ \mathrm{Log}|1/w| = -a \ \mathrm{Log}|w|.$$

Thus if $a > 0$ a source of strength a at the origin $z = 0$ is balanced by a sink of strength a at $z = \infty$.

Chapter 3

Examples of analytic functions

We now focus on various examples of complex analytic functions, starting with the rational functions, then continuing on to the exponential and logarithm functions, and finally the trigonometric functions and their inverses. Yet other examples of complex analytic functions come from the theory of ordinary differential equations.

The complex analytic functions we construct will provide important cases in which we can solve for the steady-state distribution of temperature in a given region of the plane.

3.1 Rational functions

The simplest complex analytic functions are the rational functions. These are the functions

$$R(z) = \frac{P(z)}{Q(z)}, \tag{3.1}$$

where

$$P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$$
$$\text{and} \quad Q(z) = b_n z^n + b_{n-1} z^{n-1} + \cdots + b_1 z + b_0$$

are polynomials with complex coefficients

$$a_m,\ a_{m-1},\ \ldots,\ a_1,\ a_0, \quad b_n,\ b_{n-1},\ \ldots,\ b_1,\ b_0,$$

and we make the assumptions that $a_m \neq 0$ and $b_n \neq 0$. We also assume that P and Q do not have any common factors. The function $R(z)$ is differentiable at every point of $\mathbb{C} - S$, where S is the finite set of points within $\mathbb{C}$ at which the denominator $Q(z)$ vanishes. The maximum of the two degrees, the degree of the numerator and the degree of the denominator, is called the *degree* of the rational function $R(z)$.

The sum of two rational functions is also rational function, as is the product. Indeed, one can check that with these operations of addition and multiplication, the space of rational functions satisfies the field axioms, and we denote this field by $\mathbb{C}(X)$. This is an important example of a field, which can be added to the earlier ones $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$.

The Fundamental Theorem of Algebra allows us to factor the polynomials $P(z)$ and $Q(z)$,

$$P(z) = a_m(z - z_1)^{p_1}(z - z_2)^{p_2} \cdots (z - z_r)^{p_r}$$
$$\text{and} \quad Q(z) = b_n(z - \zeta_1)^{q_1}(z - \zeta_2)^{q_2} \cdots (z - \zeta_s)^{q_s},$$

where the exponents denote the multiplicities of the roots, and this provides us with the first of the two important canonical forms for a rational function:

$$R(z) = \frac{a_m}{b_n} \frac{(z - z_1)^{p_1}(z - z_2)^{p_2} \cdots (z - z_r)^{p_r}}{(z - \zeta_1)^{q_1}(z - \zeta_2)^{q_2} \cdots (z - \zeta_s)^{q_s}}. \tag{3.2}$$

The zeros $z_1, \ldots, z_r$ of the numerator are called the *zeros* of the rational function, while the zeros $\zeta_1, \ldots, \zeta_s$ of the denominator are called its *poles*. The *order* of a zero or pole is its multiplicity as a root of either the numerator or the denominator. Poles of order one are said to be *simple*.

We say that a rational function $R(z)$ is *proper* if

$$m = \deg P(z) \leq n = \deg Q(z),$$

and *strictly proper* if $m < n$. If $R(z)$ is not strictly proper, we can divide by the denominator to obtain

$$R(z) = P_1(z) + R_1(z),$$

where $P_1(z)$ is a polynomial and the remainder $R_1(z)$ is a strictly proper rational function.

Example 1. Suppose that

$$R(z) = \frac{z^3 - 2z^2 - 7z + 21}{z^2 - 9}.$$

This rational function is not strictly proper so we can divide numerator by denominator to obtain

$$R(z) = z - 2 + R_1(z), \quad \text{where the remainder} \quad R_1(z) = \frac{2z + 3}{z^2 - 9}$$

is a strictly proper rational function.

Strictly proper rational functions have a second canonical form, called the *partial fraction decomposition*. This is extremely useful in calculating integrals of rational functions, as you may remember from calculus.

Theorem. *If the rational function $R(z) = P(z)/Q(z)$ is strictly proper, that is if $\deg P < \deg Q$, then $R(z)$ has a partial fraction decomposition*

$$
\begin{aligned}
R(z) = \frac{A_{1,0}}{(z-\zeta_1)^{q_1}} + \cdots + \frac{A_{1,q_1-1}}{(z-\zeta_1)} \\
+ \frac{A_{2,0}}{(z-\zeta_2)^{q_2}} + \cdots + \frac{A_{2,q_2-1}}{(z-\zeta_2)} + \cdots \\
+ \frac{A_{s,0}}{(z-\zeta_s)^{q_s}} + \cdots + \frac{A_{s,q_s-1}}{(z-\zeta_s)}.
\end{aligned} \tag{3.3}
$$

A proof of this theorem can be found on page 107 of Saff and Snider [10]. What you should focus on is how to calculate the partial fraction decomposition of a given rational function. Note that if all the poles of r are simple, that is all the roots of the denominator have multiplicity one, then (3.3) simplifies to

$$
R(z) = \frac{A_1}{z-\zeta_1} + \frac{A_2}{z-\zeta_2} + \cdots + \frac{A_n}{z-\zeta_n}. \tag{3.4}
$$

In this simpler case, one can find the coefficients in the partial fraction expansion by the formula

$$
A_i = \lim_{z\to\zeta_i} (z-\zeta_i)R(z). \tag{3.5}
$$

Example 2. We can write the rational function

$$
R(z) = \frac{2z+3}{z^2-9} \quad \text{as} \quad R(z) = \frac{A_1}{z-3} + \frac{A_2}{z+3},
$$

where use of (3.5) gives

$$
\begin{aligned}
A_1 = \lim_{z\to 3}(z-3)R(z) = \lim_{z\to 3}\frac{2z+3}{z+3} = \frac{9}{6} = \frac{3}{2}, \\
A_2 = \lim_{z\to -3}(z+3)R(z) = \lim_{z\to -3}\frac{2z+3}{z-3} = \frac{-3}{-6} = \frac{1}{2}.
\end{aligned}
$$

We conclude therefore that

$$
R(z) = \frac{2z+3}{z^2-9} = \frac{3/2}{z-3} + \frac{1/2}{z+3}.
$$

In the more general case, one first notes that the Taylor series expansion for $(z-\zeta_i)^{q_i}R(z)$ starts out with

$$
(z-\zeta_i)^{q_i}R(z) = A_{i,0} + A_{i,1}(z-\eta_i) + \cdots + A_{i,j}(z-\eta_i)^j + \cdots ,
$$

so the coefficient $A_{i,j}$ is given by the formula from calculus for the coefficient in this Taylor expansion,

$$
A_{i,j} = \lim_{z\to\zeta_i}\left(\frac{1}{j!}\frac{d^j}{dz^j}\left[(z-\zeta_i)^{q_i}R(z)\right]\right). \tag{3.6}
$$

Example 3. We can write the rational function

$$R(z) = \frac{z^2+2z+3}{(z-3)^3} \quad \text{as} \quad R(z) = \frac{A_{1,0}}{(z-3)^3} + \frac{A_{1,1}}{(z-3)^2} + \frac{A_{1,2}}{(z-3)},$$

where it follows from (3.6) that

$$\begin{aligned}
A_{1,0} &= \lim_{z\to 3}(z-3)^3 R(z) = \lim_{z\to 3}(z^2+2z+3) = 18,\\
A_{1,1} &= \lim_{z\to 3}\left(\frac{d}{dz}(z-3)^3 R(z)\right) = \lim_{z\to 3}\left(\frac{d}{dz}(z^2+2z+3)\right) = \lim_{z\to 3}(2z+2) = 8,\\
A_{1,2} &= \lim_{z\to 3}\left(\frac{1}{2}\frac{d^2}{dz^2}(z-3)^3 R(z)\right) = \lim_{z\to 3}\left(\frac{1}{2}\frac{d^2}{dz^2}(z^2+2z+3)\right) = \lim_{z\to 3}(1) = 1.
\end{aligned}$$

We conclude therefore that

$$R(z) = \frac{z^2+2z+3}{(z-3)^3} = \frac{18}{(z-3)^3} + \frac{8}{(z-3)^2} + \frac{1}{(z-3)}.$$

It is quite convenient to regard the argument z of the rational functions $R(z)$, as well as the values of $R(z)$. as ranging over the extended complex plane $\mathbb{C}\cup\{\infty\}$. In considering the extension to $\mathbb{C}\cup\{\infty\}$, we make use of a new variable $w = 1/z$ which is well-behaved near ∞. Suppose that the numerator and denominator in the rational function have the same degree,

$$R(z) = \frac{a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0}{b_n z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0}.$$

Then replacing z by $1/w$ yields

$$\begin{aligned}
R(z) &= \frac{a_n(1/w)^n + a_{n-1}(1/w)^{n-1} + \cdots + a_1(1/w) + a_0}{b_n(1/w)^n + b_{n-1}(1/w)^{n-1} + \cdots + b_1(1/w) + b_0}\\
&= \frac{a_n + a_{n-1}w + \cdots + a_1 w^{n-1} + a_0 w^n}{b_n + b_{n-1}w + \cdots + b_1 w^{n-1} + b_0 w^n}.
\end{aligned}$$

Setting $z = \infty$ is the same as setting $w = 0$, so we let R denote also the extension of the rational function R to $\mathbb{C}\cup\{\infty\}$,

$$R(\infty) = \frac{a_n}{b_n}.$$

More generally, the reader can easily check that if

$$R(z) = \frac{P(z)}{Q(z)}, \quad \text{with} \quad \deg P = m \quad \text{and} \quad \deg Q = n,$$

then R has a zero of order $n-m$ at ∞ if $m < n$ and a pole of order $m-n$ at ∞ if $m > n$. In particular, strictly proper rational functions have zeros at ∞.

Similarly, we regard the value of $R(z)$ at a pole ζ_i as ∞. With this convention we can regard R as a map

$$R : \mathbb{C} \cup \{\infty\} \longrightarrow \mathbb{C} \cup \{\infty\}. \tag{3.7}$$

It is often helpful to use stereographic projection to regard this as defining a map from S^2 to itself. It turns out that if R has degree n there will be n solutions to the equation $R(z) = c$ for most choices of $c \in \mathbb{C} \cup \{\infty\}$.

Example 4. Let us reconsider the rational function of Example 1,

$$R(z) = \frac{z^3 - 2z^2 - 7z + 21}{z^2 - 9} = z - 2 + \frac{2z + 3}{z^2 - 9}.$$

As in Example 2, we can also write

$$R(z) = z - 2 + \frac{3/2}{z - 3} + \frac{1/2}{z + 3},$$

so R has simple poles at 3 and -3. Replacing z by $1/w$ yields,

$$R(z) = \frac{1}{w} - 2 + w\frac{2 + 3w}{1 - 9w^2},$$

so R also has a simple pole at ∞. Thus there are exactly three elements of $\mathbb{C} \cup \{\infty\}$ which solve the equation $R(z) = \infty$, namely 3, -3 and ∞.

3.2 Linear fractional transformations

Of central importance among the rational functions are the *linear fractional transformations* or *Möbius transformations*, the rational functions of degree one:

$$w = T(z) = \frac{az + b}{cz + d}, \tag{3.8}$$

where a, b, c and d are complex numbers such that $ad - bc \neq 0$. By solving for z as a function of w, we can show that any linear fractional transformation T has an inverse,

$$z = T^{-1}(w) = \frac{dw - b}{-cw + a},$$

as one verifies by a calculation. Thus linear fractional transformations are always one-to-one and onto as maps from $\mathbb{C} \cup \{\infty\}$ to itself. Indeed, with a little effort, it can be shown that they are the only complex analytic functions with complex analytic inverses from $\mathbb{C} \cup \{\infty\}$ to itself.

Moreover, one easily checks that the composition of two linear fractional transformations is another linear fractional transformation. Indeed, composition makes the space of linear fractional transformations into what algebraists call a group.

When $c = 0$ and $d = 1$, the linear fractional transformation reduces to a *complex linear transformation*

$$z \mapsto w = T(z) = az + b.$$

The complex linear transformations include the

$$\begin{cases} \text{translations} & z \mapsto w = T(z) = z + b, \\ \text{rotations} & z \mapsto w = T(z) = e^{i\theta} z, \\ \text{and expansions and contractions} & z \mapsto w = T(z) = \rho z, \end{cases}$$

where θ and ρ are real numbers, ρ being nonzero. When $a = d = 0$ and $b = c = 1$, the linear fractional transformation reduces to an *inversion*

$$w = T(z) = \frac{1}{z}.$$

Here T is a reflection in the circle

$$z \mapsto \frac{1}{\bar{z}}, \quad re^{i\theta} \mapsto \frac{1}{r} e^{i\theta},$$

followed by conjugation.

The following Proposition is quite useful in determining the properties of a given linear fractional transformation. Linear fractional transformations will become increasingly important as the course progresses, and will be studied in more detail in 122B; see §7.3 of [10] or Chapter 7, § 5 of [7].

Proposition. *a. Any linear fractional transformation is the composition of complex linear maps and inversions. b. Any linear fractional transformation takes circles and lines to circles and lines. c. Given any three distinct points z_1, z_2 and z_3 of $\mathbb{C} \cup \{\infty\}$, there is a unique linear fractional transformation T such that*

$$T(z_1) = 0, \quad T(z_2) = 1 \quad \text{and} \quad T(z_3) = \infty. \tag{3.9}$$

Sketch of proof of a: Note that the transformation (3.8) is unchanged if we make the replacements

$$a \mapsto \lambda a, \quad b \mapsto \lambda b, \quad c \mapsto \lambda c, \quad d \mapsto \lambda d,$$

where $\lambda \neq 0$. So we can assume without loss of generality that $ad - bc = 1$. Moreover, we can assume without generality that $c \neq 0$, because when $c = 0$, the transformation is clearly linear. We can then factor the map

$$z' = T(z) = \frac{az + b}{cz + d}$$

into a composition of four transformations

$$z_1 = z + \frac{d}{c}, \quad z_2 = c^2 z_1, \quad z_3 = -\frac{1}{z_2}, \quad z' = z_3 + \frac{a}{c}. \tag{3.10}$$

To see this, note that

$$z_2 = c(cz+d), \quad z_3 = -\frac{1}{c(cz+d)},$$

$$z' = \frac{a(cz+d)}{c(cz+d)} - \frac{1}{c(cz+d)} = \frac{acz+ad-1}{c(cz+d)} = \frac{az+b}{cz+d}.$$

Each of the transformations in (3.10) is either complex linear or an inversion, so the Proposition is proven.

Sketch of proof of b: Complex linear transformations take circles and lines to circles and lines, so we need only check that inversions take circles and lines to circles and lines. But the equation of a circle can be written as

$$A(x^2+y^2) + \mathrm{Re}[(C-iD)(x+iy)] + B = 0,$$

for some choice of real constants A, B, C and D. We can rewrite this as

$$Az\bar{z} + \mathrm{Re}[(C-iD)z] + B = 0.$$

If we replace z by $1/w$, this becomes

$$\frac{A}{w\bar{w}} + \mathrm{Re}\left[\frac{(C-iD)}{w}\right] + B = 0$$

or

$$A + \mathrm{Re}\left[(C-iD)\bar{w}\right] + Bw\bar{w} = 0,$$

which is once again the equation for a circle.

Sketch of proof of c: If none of the three points is infinity, the linear fractional transformation satisfying (3.9) is

$$T(z) = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}. \tag{3.11}$$

One needs to adjust this formula appropriately when one of the three points is ∞.

Example. Suppose that we want a linear fractional transformation T such that

$$T(-i) = 0, \quad T(1) = 1, \quad T(i) = \infty.$$

Use of the formula (3.11) yields

$$T(z) = \frac{(z+i)(1-i)}{(z-i)(1+i)} = \frac{z+i}{iz+1}.$$

This transformation takes the unique circle through $-i$, 1 and i, which turns out to be the unit circle, to the straight line through 0, 1 and ∞, which is the x-axis. Moreover, $T(0) = i$. From this one can conclude that T takes the unit disk

$$D = \{z \in \mathbb{C} : |z| < 1|]$$

to the upper half-plane

$$H = \{x+iy \in \mathbb{C} : y > 0\}.$$

3.3 Exponential and trigonometric functions

Earlier, we defined the complex exponential function in terms of real-valued functions as

$$w = \exp(z) = e^z = e^x(\cos y + i \sin y), \tag{3.12}$$

and used identities from trigonometry to show that

$$e^{z_1+z_2} = e^{z_1}e^{z_2},$$

as well as to show that the exponential function is periodic of period $2\pi i$:

$$\exp(z + 2\pi k i) = \exp(z), \quad \text{for } k \in \mathbb{Z}. \tag{3.13}$$

Moreover, writing

$$e^z = u(x,y) + iv(x,y), \quad \text{where} \quad u(x,y) = e^x \cos y \quad \text{and} \quad v(x,y) = e^x \sin y$$

and using the Cauchy-Riemann equations, we find that

$$\frac{d}{dz}(e^z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = e^x \cos y + ie^x \sin y = e^z. \tag{3.14}$$

Although the exponential function exp is certainly quite well-behaved on $\mathbb{C}$, it does not extend to a well-behaved function on the extended complex plane $\mathbb{C} \cup \{\infty\}$. Neither

$$\lim_{w\to 0} e^{1/w} \quad \text{nor} \quad \lim_{w\to 0} \frac{1}{e^{1/w}}$$

exists, and hence one says that $\exp(z)$ has an "essential singularity" at ∞. Indeed, one can show that $e^{1/w}$ takes on every nonzero complex number infinitely many times when w lies in any neighborhood $N(0;\varepsilon)$ of 0.

We would like to define a function which is inverse to exp on $\mathbb{C} - \{0\}$, but the periodicity makes it impossible to define an inverse which is a genuine function. Indeed, if $w = e^z$, it only follows from (3.12) that

$$w = re^{i\theta} = r(\cos\theta + i\sin\theta) = e^x(\cos y + i \sin y),$$

so y is one of the many possible values of the angular coordinate θ, all of these values differing by integer multiples of 2π. Thus we write

$$x = \mathrm{Log}(r) = \mathrm{Log}(|w|), \quad y = \theta = \arg(w),$$

where arg is the multiple-valued function which gives the various possible angular coordinates θ of w, and

$$z = x + iy = \mathrm{Log}(|w|) + i\arg(w).$$

In this way, we are led to define the multiple-valued function log by

$$\log(w) = \mathrm{Log}(|w|) + i\arg(w),$$

where $\mathrm{Log}(|w|)$ is the usual natural logarithm of the nonzero real number $|w|$. It follows from (3.13) that the various values of the multiple-valued function log differ by integer multiples of 2π.

Dangerous curve: It is important to note that a multiple-valued function is NOT a genuine function as studied in the rest of mathematics. A genuine function can only assume one value for a given choice of argument. Nevertheless, the notion of multiple-valued function is commonly used within the theory of complex variables, because it is quite useful. In more advanced treatments of complex variables, one tries to eliminate multiple-valued functions on open subsets of the complex plane by passing instead to genuine single-valued functions defined on "Riemann surfaces" over the open subsets (see [3]).

As we saw earlier, we can eliminate the ambiguity in the logarithm by replacing $\mathbb{C} - \{0\}$ by the smaller domain

$$D = \mathbb{C} - \{x + iy \in \mathbb{C} : x \leq 0\}, \tag{3.15}$$

and choosing a *branch* of the muliple-valued function log that is single-valued in D. We do this by first letting $\mathrm{Arg}(w)$ be the value of the angular coordinate θ which lies in the interval $(-\pi, \pi]$ and then define

$$\mathrm{Log} : D \longrightarrow \mathbb{C} \quad \text{by} \quad \mathrm{Log}(w) = \mathrm{Log}(|w|) = i\mathrm{Arg}(w).$$

Choosing a branch does give us a genuine single-valued complex analytic logarithm function, and as we saw when discussing the Cauchy-Riemann equations,

$$\frac{d}{dw}\mathrm{Log}(w) = \frac{1}{w}.$$

However, this destroys some of the nice properties one might hope for the logarithm; for example, although

$$\log(z_1 z_2) = \log(z_1) + \log(z_2),$$

where equality means that the values taken by the multiple-valued functions on the two sides are the same, it is not true that

$$\mathrm{Log}(z_1 z_2) = \mathrm{Log}(z_1) + \mathrm{Log}(z_2)$$

in general—the formula is sometimes off by an integer multiple of $2\pi i$.

One of the most useful applications of the multiple-valued logarithm function is in defining arbitrary powers of a complex number. The definition is motivated by the fact that

$$z^n = \exp\left(n\log(z)\right), \quad \text{when} \quad n \in \mathbb{N}.$$

Definition. If $z \in \mathbb{C} - \{0\}$ and $\alpha \in \mathbb{C}$, we define z^α by

$$z^\alpha = \exp\left(\alpha\log(z)\right).$$

Example 1. Let us consider the possible values of

$$i^{-2i} = \exp(-2i\log(i)).$$

To do this, we first note that

$$\log(i) = \log 1 + i\arg(i) = 0 + i\frac{\pi}{2} + 2\pi ki, \quad \text{for } k \in \mathbb{Z},$$

so

$$i^{-2i} = \exp(\pi + 4\pi k), \quad \text{for } k \in \mathbb{Z}.$$

This example illustrates that just like logarithms, the complex power z^α can have infinitely many different values. Although square roots can have only two different values, they provide important examples of multiple-valued functions. Later, when we want to integrate square roots along contours, we will need to choose a particular branch for the integrals to be well-defined. We provide a few examples to illustrate how this is done:

Example 2. We can construct two branches f_1 and f_2 of the square root function

$$w = \pm\sqrt{z},$$

by setting

$$f_1(z) = \exp\left(\frac{1}{2}\mathrm{Log}(z)\right), \qquad f_2(z) = -\exp\left(\frac{1}{2}\mathrm{Log}(z)\right),$$

for $z \in D$, where D is the domain defined by (3.15), the complex plane minus the negative x-axis. One can imagine two copies D_1 and D_2 of D (called *sheets*) which are glued together along the negative x-axis. As one crosses the negative x-axis one passes from sheet to the other. The union of the two sheets can be regarded as a "Riemann surface" on which the square root function is a genuine single-valued function.

Example 3. Suppose that we want to construct a branch of the multiple-valued function

$$w = \pm\sqrt{1 - z^2}. \tag{3.16}$$

To do this, we note that the multiple-valued function can be defined by

$$w = \exp\left(\frac{1}{2}\log(1 - z^2)\right),$$

and that $\log(1 - z^2)$ fails to be defined when $z = \pm 1$. We can then construct two branches

$$w = f_1(z) = \exp\left(\frac{1}{2}\mathrm{Log}\left(1 - z^2\right)\right)$$

and

$$w = f_2(z) = -\exp\left(\frac{1}{2}\mathrm{Log}\left(1 - z^2\right)\right),$$

which will be a single-valued functions defined so long as $1 - z^2$ does not lie on the negative part of the x-axis, that is, so long as z is not real with $|z| > 1$. Thus we can regard $f - z$ and f_2 as analytic functions defined on the open set

$$\begin{aligned} U = \mathbb{C} - \{ z = x + iy \in \mathbb{C} : \; y = 0 \text{ and } x \geq 1 \text{ or } x \leq -1 \; \} \\ = \mathbb{C} - ((-\infty, -1] \cup [1, \infty)). \end{aligned}$$

One can imagine that the two branches of the multiple-valued function $\pm\sqrt{1 - z^2}$ are interchanged when one crosses the subset $(-\infty, -1) \cup (1, \infty)$ of $\mathbb{R}$.

Example 4. Similarly, we can construct branches of the multiple-valued function

$$w = \pm\sqrt{z^2 - 1} = \pm i\sqrt{1 - z^2},$$

which is not much different from the function considered in the previous example, but this time we seek branches are defined and single-valued outside the unit disk. For this, we note first that the multiple-valued function can also be written as

$$w = z \; \exp\left(\frac{1}{2} \log \left(1 - \frac{1}{z^2} \right) \right),$$

and in this representation $1 - (1/z)^2$ lies on the negative x-axis exactly when z is real and $|z| < 1$. Thus the functions

$$w = f_1(z) = z \; \exp\left(\frac{1}{2} \mathrm{Log} \left(1 - \frac{1}{z^2} \right) \right)$$

and

$$w = f_2(z) = -z \; \exp\left(\frac{1}{2} \mathrm{Log} \left(1 - \frac{1}{z^2} \right) \right)$$

will be a single-valued functions defined on the set

$$U = \mathbb{C} - \{ z = x + iy \in \mathbb{C} : y = 0 \; \& \; -1 \leq x \leq 1 \} = \mathbb{C} - [-1, 1].$$

In this case, one can imagine that the two branches of the multiple-valued function $\pm\sqrt{z^2 - 1}$ are interchanged when one crosses the interval $(-1, 1) \subseteq \mathbb{R}$.

Trigonometric functions. As mentioned before, we can use Euler's identity to express the trigonometric functions cosine and sine in terms of exponentials,

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i},$$

thereby achieving a simplifying unity between exponential and trigonometric functions. We can use these expressions to determine all of the usual properties of the trigonometric functions. For example, it follows from (3.14) that

$$\frac{d}{dz}(\sin z) = \cos z, \qquad \frac{d}{dz}(\cos z) = -\sin z.$$

Thus, it follows from the definition of derivative that if $f(z) = \sin z$,

$$1 = \cos(0) = f'(0) = \lim_{\Delta z \to 0} \frac{f(\Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\sin(\Delta z)}{\Delta z} = \lim_{h \to 0} \frac{\sin h}{h},$$

a limit familiar in first year calculus when h is assumed to be real.

Using the logarithm, we can also solve to find the inverses of the trigonometric functions. Thus if

$$z = \sin w = \frac{e^{iw} - e^{-iw}}{2i},$$

we can multiply through by e^{iw}, obtaining

$$2ie^{iw}z = \left(e^{iw}\right)^2 - 1 \quad \text{or} \quad \left(e^{iw}\right)^2 - 2ize^{iw} - 1 = 0.$$

Thus it follows from the quadratic formula that

$$e^{iw} = iz \pm \sqrt{1 - z^2},$$

and we find that

$$w = \arcsin(z) = -i \log\left(iz \pm \sqrt{1 - z^2}\right), \tag{3.17}$$

which is, of course, a multiple-valued function. Similarly, the inverse of the cosine is a mutiple-valued function

$$w = \arccos(z) = \frac{\pi}{2} - \arcsin(z) = \frac{\pi}{2} - i \log\left(iz \pm \sqrt{1 - z^2}\right).$$

To choose a specific branch of the inverse trigonometric functions, we must choose a branch of the square root as well as of the logarithm. For any such branch, it follows from (3.17) that

$$\frac{d}{dz}(\arcsin(z)) = \frac{1}{\pm\sqrt{1 - z^2}}. \tag{3.18}$$

Note that by Example 3, a single-valued branch of the derivative of the inverse of the sine can be defined over

$$U = \mathbb{C} - ((-\infty, -1] \cup [1, \infty)),$$

and we will later see that this enables us to construct a single-valued inverse of the sine itself over the same region U.

Similarly, we can find the inverses of the hyperbolic cosine; thus if

$$z = \cosh w = \frac{e^w + e^{-w}}{2},$$

we can multiply through by e^w, obtaining

$$2e^w z = (e^w)^2 + 1 \quad \text{or} \quad (e^w)^2 - 2ze^w + 1 = 0.$$

This time the quadratic formula, yields

$$e^w = \frac{2z \pm \sqrt{4z^2 - 4}}{2} = z \pm \sqrt{z^2 - 1}.$$

Thus we find that

$$w = \operatorname{arccosh}(z) = \log\left(z \pm \sqrt{z^2 - 1}\right) \tag{3.19}$$

Differentiation this time yields

$$\frac{d}{dz}(\operatorname{arccosh}(z)) = \frac{1}{\pm\sqrt{z^2 - 1}},$$

the derivative possessing a single-valued branch over the domain $\mathbb{C} - [-1, 1]$ by Example 4.

3.4 Steady-state temperature distributions

We now illustrate the use of the real and imaginary parts of the complex logarithm in determining the steady-state temperature distribution in several regions of the plane.

But first we must show that steady-state temperature is represented by a solution to the Laplace equation. We suppose that a region U of the (x, y)-plane contains a material that has homogeneous properties. We would like to determine the steady-state temperature within the region when the temperature is given on the boundary.

Let $u(x, y)$ be the temperature at the point (x, y). Then the flow of heat at (x, y) should be $\mathbf{V}(x, y)$, where

$$\mathbf{V}(x, y) = -(\kappa \nabla u)(x, y) = -\kappa \frac{\partial u}{\partial x}(x, y)\mathbf{i} - \kappa \frac{\partial u}{\partial x}(x, y)\mathbf{j}, \tag{3.20}$$

where κ is a positive constant, called the *thermal conductivity* of the material. We agree to set

$$\mathbf{V}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}.$$

If D is a region within U with smooth boundary ∂D, then it follows from the divergence theorem that

$$\int_{\partial D} (M\mathbf{i} + N\mathbf{j}) \cdot \mathbf{N} ds = \int\int_D \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy,$$

where $\mathbf{N}$ is the outward-pointing unit normal to ∂D, and s is the arc-length parameter along ∂D. Recall that the line integral on the left is calculated by means of a unit-speed parametrization $\gamma : [a, b] \to \partial D$ with $\gamma(a) = \gamma(b)$.

This line integral can be interpreted as the rate at which heat is flowing outward across ∂D. Thus if no heat is being created or destroyed within D, the line integral must vanish, and

$$\int\int_D \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy = 0.$$

If no heat is being created or destroyed anywhere within U, this double integral must vanish for every choice of $D \subseteq U$, and this can only happen if

$$\nabla \cdot \mathbf{V} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \equiv 0 \quad \text{on } U.$$

In the special case where $\mathbf{V}$ is given by (3.20), we conclude that the temperature u must satisfy Laplace's equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

or equivalently, u must be a harmonic function. (See §2.6 of [10] for a more leisurely discussion.)

Since harmonic functions are the real parts of complex analytic functions, we have a nice application of complex analytic functions: Their real parts can represent the steady-state temperature distribution in a homogeneous medium.

Example 1. Suppose we want to determine the steady state temperature distribution in the annular region

$$D = \{x + iy \in \mathbb{C} : 1 \le x^2 + y^2 \le e^2\},$$

given that the temperature on the circle $x^2 + y^2 = 1$ is 40 degrees and the temperature on the circle $x^2 + y^2 = e^2$ is 70 degrees.

To solve this problem, it is quite natural to use polar coordinates (r, θ). We need to find those harmonic functions which depend only on the radial coordinate r. This problem was solved in the discussion following Exercise C in §2.4, and we found that the only such harmonic functions are

$$u = a \text{ Log } r + b = a \text{ Log}|z| + b,$$

where a and b are real constants. The boundary conditions are that $u = 40$ when $r = 1$ and $u = 70$ when $r = e$. Thus

$$\begin{cases} a \text{ Log}(1) + b = b = 40, \\ a \text{ Log}(e) + b = a + b = 70, \end{cases}$$

and we find that $b = 40$ and $a = 30$, so

$$u = 30 \text{ Log } r + 40.$$

Example 2. Suppose we want to determine the steady state temperature distribution $u(z) = u(x, y)$ in the wedge

$$D = \{x + iy \in \mathbb{C} : x > 0,\ y < x,\ y > -x\},$$

given that the temperature on the ray $y = x,\ x > 0$ is 90 degrees and the temperature on the ray $y = -x,\ x > 0$ is 50 degrees. Once again it is natural to use polar coordinates (r, θ), and we seek those harmonic functions which depend only on the angular coordinate θ. This time the only such harmonic functions are

$$u = a\,\theta + b,$$

where a and b are real constants. The boundary conditions are that $u = 90$ when $\theta = \pi/4$ and $u = 50$ when $\theta = -\pi/4$. Thus

$$\begin{cases} a(\pi/4) + b = 90, \\ -a(\pi/4) + b = 50, \end{cases}$$

and we find that the solution to the linear equations is $a = 80/\pi$ and $b = 70$, so

$$u = \frac{80\,\theta}{\pi} + 70 \quad \text{or} \quad u = \frac{80}{\pi}\mathrm{Arg}(x + iy) + 70.$$

Example 3. Suppose we want to determine the steady state temperature distribution $u(z) = u(x, y)$ in the upper half-plane, given that the temperature along the real axis is given by

$$u(x, 0) = \begin{cases} 90, & \text{for } x < 1, \\ 70, & \text{for } -1 < x < 1, \\ 50, & \text{for } x > 1. \end{cases}$$

In this case, it is natural to consider the temperature as given by a superposition of argument functions

$$u(z) = a + b_1\mathrm{Arg}(z - 1) + b_2\mathrm{Arg}(z + 1).$$

The boundary conditions in this case give

$$a + \pi b_1 + \pi b_2 = 90, \quad a + \pi b_1 = 70, \quad a = 50,$$

and we quickly find that the solution is

$$u(z) = 50 + \frac{20}{\pi}\mathrm{Arg}(z - 1) + \frac{20}{\pi}\mathrm{Arg}(z + 1).$$

Proposition. *Suppose that U and V are open subsets of $\mathbb{C}$. If $f : U \to V$ is complex analytic, and $u : V \to \mathbb{R}$ is harmonic, then $u \circ f : U \to \mathbb{R}$ is harmonic.*

Sketch of proof: First reduce to the case where U and V are open balls. Then one can use the Poincaré Lemma to construct a harmonic conjugate v to u and

$$g = u + iv : V \to \mathbb{C}$$

is complex analytic. Then it follows from the chain rule that $g \circ f : U \to \mathbb{C}$ is also complex analytic, and hence

$$u \circ f = \mathrm{Re}(g \circ f) : U \to \mathbb{R}$$

is harmonic.

Example 4. We can now use the preceding example together with the linear fractional transformation

$$T(z) = \frac{z + i}{iz + 1},$$

described at the end of §3.2 to determine the steady-state temperature in the unit disk,

$$D = \{x + iy \in \mathbb{C} : x^2 + y^2 \leq 1\}$$

which satisfies the boundary conditions that the temperature be 50 degrees on the part C_1 of the circle $x^2 + y^2 = 1$ lying in the first quadrant, 70 degrees on the part C_2 of the circle $x^2 + y^2$ lying below the x-axis, and 90 degrees on the remaining quarter circle C_3.

It follows from the Proposition that if $u(z)$ is the harmonic function of Example 3, then $u \circ T$ is also harmonic. Moreover, T takes D to the upper half-plane, takes the counterclockwise unit circle to the x-axis traversed in the positive direction, and satisfies

$$T(-1) = -1, \quad T(1) = 1, \quad T(i) = \infty.$$

Hence it takes C_1, C_2 and C_3 to the segments $x > 1$, $-1 < x < 1$ and $x < -1$ along the x-axis, and the desired function is

$$(u \circ T)(z) = 50 + \frac{20}{\pi}\mathrm{Arg}(T(z) - 1) + \frac{20}{\pi}\mathrm{Arg}(T(z) + 1).$$

Chapter 4

Contour integrals

We now turn to the question of integrating complex differentials along curves in the complex plane $\mathbb{C}$. We can think of this as an extension of real valued integrals such as

$$\int_a^b f(x)dx$$

to the case where f and x are allowed to take on complex values.

A complex differential is an expression of the form

$$f(z)dz, \quad \text{where} \quad f(x+iy) = u(x,y) + iv(x,y) \quad \text{and} \quad dz = dx + idy.$$

We can write a complex differential in terms of real and imaginary parts as

$$\begin{aligned} f(z)dz &= (u(x,y) + iv(x,y))(dx + idy) \\ &= [u(x,y)dx - v(x,y)dy] + i\,[v(x,y)dx + u(x,y)dy], \end{aligned}$$

and it follows from the Cauchy-Riemann equations that the coefficient $f(z)$ is holomorphic if and only if the real and imaginary parts both represent incompressible and irrotational fluid flow. In this chapter, we will show how to construct the contour integral of a complex differential, which is just the line integral as studied in several variable calculus.

Our main goal will be to prove Cauchy's theorem, which states that the integral of a holomorphic differential defined in the entire complex plane around a closed contour is zero, and we will see that it is just a consequence of Green's Theorem from several variable calculus, at least when $f(z)$ has a continuous complex derivative. Goursat found a proof of Cauchy's theorem that requires only that f be continuous and that f has a complex derivative at every point; in his proof no assumption was needed about the continuity of the complex derivative. We will see that Cauchy's theorem then implies that complex analytic functions have continuous derivatives of all orders, and gives explicit formulae for these derivatives in terms of contour integrals. Moreover, Cauchy's theorem has many remarkable applications, including a proof of the Fundamental Theorem of Algebra, that a nonconstant polynomial with complex coefficients always has at least one complex root.

4.1 Contours

We will start by reviewing the notion of line integral, or *contour integral* according to the terminology most frequently used in complex analysis. To do this, we first need to discuss the smooth parametrization of curves.

Definition 1. A *smooth parametrized arc* in $\mathbb{C}$ is a continuously differentiable map $z : [a, b] \to \mathbb{C}$, for some choice of closed interval $[a, b] \subseteq \mathbb{R}$ with $a < b$, such that $z'(t)$ is never zero, where

$$z(t) = x(t) + iy(t) \quad \text{and} \quad z'(t) = \frac{dx}{dt}(t) + i\frac{dy}{dt}(t).$$

We will say that $z(a)$ is the *initial point*, and $z(b)$ the *terminal point*, of the smooth parametrized arc.

Example 1. The simplest case of a smooth parametrized arc is the constant speed straight line segment from a point $z_0 \in \mathbb{C}$ to $z_1 \in \mathbb{C}$, where $z_0 \neq z_1$, which starts at z_0 at time $t = 0$ and ends at z_1 at time $t = 1$:

$$z : [0, 1] \to \mathbb{C} \quad \text{where} \quad z(t) = (1 - t)z_0 + tz_1.$$

For example, the straight line segment C from $2 + 5i$ to $3 + 7i$ is parametrized by

$$z : [0, 1] \to \mathbb{C} \quad \text{where} \quad z(t) = (1 - t)(2 + 5i) + t(3 + 7i),$$

with $2 + 5i$ being the initial point of C and $3 + 7i$ the terminal point.

Given a smooth parametrized arc $z : [a, b] \to \mathbb{C}$, a *reparametrization* of z is a map

$$z \circ \tau : [c, d] \to \mathbb{C}, \quad \text{where} \quad \tau : [c, d] \to [a, b]$$

is an increasing continuously differentiable map with differentiable inverse. Two parametrized arcs in $\mathbb{C}$ are *equivalent* if they are related by reparametrization. Finally, by a *smooth arc* C in $\mathbb{C}$ we mean an equivalence class of smooth parametrized arcs.

Example 2. The *catenary* is the smooth arc C in $\mathbb{C}$ formed by a chain hanging between two endpoints, and defined by the equation

$$y = \cosh x, \quad \text{where} \quad \cosh x = \frac{e^x + e^{-x}}{2}.$$

The simplest parametrization is obtained by letting $x = t$, so $y = \cosh t$:

$$z : [-1, 1] \to \mathbb{C} \quad \text{such that} \quad z(t) = t + (\cosh t)i.$$

If one thinks of the parameter t as time, then the total distance traversed is just the integral of the speed:

$$s = (\text{arc length}) = \int_0^t |z'(\tau)|\, d\tau.$$

But since the derivative of $\cosh t$ is $\sinh t$,

$$z'(t) = 1 + (\sinh t)i \quad \Rightarrow \quad |z'(t)| = \sqrt{1 + \sinh^2 t} = \cosh t,$$

so the arc length is

$$s = \sigma(t) = \int_0^t |z'(\tau)|\, d\tau = \int_0^t \cosh \tau d\tau = \sinh t.$$

But it follows from (3.19) that the hyperbolic sine has an inverse given by

$$t = \tau(s) = \operatorname{arcsin}(s) = \operatorname{Log}\left(s + \sqrt{s^2 + 1}\right).$$

One thereby obtains a reparametrization

$$z \circ \tau : [\sinh(-1), \sinh(1)] \to \mathbb{C}$$

of the catenary by arc length with the explicit formula

$$z \circ \tau(s) = \operatorname{Log}\left(s + \sqrt{s^2 + 1}\right) + \left(\sqrt{s^2 + 1}\right) i.$$

One can check directly that $z \circ \tau$ has unit speed, in other words that

$$\left| \frac{d(z \circ \tau)}{ds}(s) \right| = 1.$$

Usually, one of the smooth parametrized arcs z defining a smooth arc C will be one-to-one and we can identify the equivalence class of parametrization with the image of that parametrization z. This allows us to think identify C with the image of any of its parametrizations, a subset of $\mathbb{C}$. In fact, Saff and Snider require that z be one-to-one as part of the definition of smooth arc (see §4.1 of [10]), but it is sometimes convenient to have smooth arcs which are not one-to-one. We will often say that z is a *parametrization* of the smooth arc C. The parametrization gives a *direction* to C, the direction of increasing time. By $-C$, we will mean the arc with the same image, but traversed in the opposite direction.

Example 3. If C is the directed line segment from $4 + i$ to $5 + 2i$ which is parametrized by

$$z : [0, 1] \to \mathbb{C} \quad \text{where} \quad z(t) = (1 - t)(4 + i) + t(5 + 2i),$$

then

$$z : [0, 1] \to \mathbb{C} \quad \text{where} \quad z(t) = (1 - t)(5 + 2i) + t(4 + i)$$

is a parametrization of $-C$, which has the initial and terminal points reversed.

Definition 2. A *smooth closed parametrized curve* in $\mathbb{C}$ is a smooth map $z : [a, b] \to \mathbb{C}$ with $a < b$ such that $z'(t)$ is never zero,

$$z(b) = z(a) \quad \text{and} \qquad z'(b) = z'(a).$$

We will say that $z(a) = z(b)$ is both the *initial point* and the *terminal point* of the smooth closed parametrized curve.

Given a smooth parametrized arc $z : [a, b] \to \mathbb{C}$, a *reparametrization* of z is a map

$$z \circ \tau : [c, d] \to \mathbb{C} \quad \text{where} \quad \tau : [c, d] \to [a, b]$$

is an increasing differentiable map with differentiable inverse such that

$$(z \circ \tau)'(c) = (z \circ \tau)'(d).$$

As in the case of smooth arcs, we can define a *smooth closed curve* in $\mathbb{C}$ to be an equivalence class of smooth closed parametrized curves, two being equivalent if they differ by a parametrization. Any one of the parametrizations of C gives a *direction* to C, the direction of increasing time, and by $-C$, we will mean the smooth closed curve with the same image, but traversed in the opposite direction.

Example 4. The simplest case of a smooth closed curve is the circle of radius ρ centered at a point $z_0 \in \mathbb{C}$. The counterclockwise parametrization of this circle is given by

$$z : [0, 2\pi] \to \mathbb{C} \quad \text{where} \quad z(t) = z_0 + \rho e^{it}, \tag{4.1}$$

while the clockwise parametrization is

$$z : [0, 2\pi] \to \mathbb{C} \quad \text{where} \quad z(t) = z_0 + \rho e^{-it}.$$

Thus, for example, the counterclockwise circle C of radius 7 centered at $2 + 5i$ is parametrized by

$$z : [0, 2\pi] \to \mathbb{C} \quad \text{where} \quad z(t) = 2 + 5i + 7e^{it}.$$

By a *smooth directed curve* C we mean either a smooth arc or a smooth closed curve with a choice of direction.

Definition 3. A *smooth contour* Γ in $\mathbb{C}$ will be a finite sequence $(C_1, \ldots, C_n)$ of smooth directed curves such that the terminal point of C_i is the initial point of C_{i+1}, for $1 \le i \le n - 1$. In this case, we will usually write

$$\Gamma = C_1 + C_2 + \cdots + C_n,$$

noting that the order in the sum is important. The *initial point* of Γ is the initial point of C_1 and the *terminal point* of Γ is the terminal point of C_n.

Contours are also sometimes called *piecewise smooth curves*. Indeed, if we choose parametrizations

$$z_i : [a_i, b_i] \to \mathbb{C}, \quad \text{for } 1 \le i \le n,$$

such that $b_i = a_{i+1}$, for $1 \le i \le n-1$, these parametrization piece together to give a *piecewise smooth* parametrization

$$z : [a_1, b_n] \to \mathbb{C} \quad \text{of} \quad \Gamma.$$

Note that given a contour

$$\Gamma = C_1 + C_2 + \cdots + C_n, \quad \text{we can set} \quad -\Gamma = -C_n - \cdots - C_1,$$

and if

$$\Gamma_1 = C_1 + \cdots + C_n \quad \text{and} \quad \Gamma_2 = C_{n+1} + \cdots + C_{n+m}$$

are smooth contours with the terminal point of Γ_1 equalling the initial point of Γ_2, we can form the sum

$$\Gamma_1 + \Gamma_2 = C_1 + \cdots + C_n + C_{n+1} + \cdots + C_{n+m}.$$

We say that a smooth contour $\Gamma = C_1 + \cdots + C_n$ is *polygonal* if each of the C_i's is a straight line segment.

Finally, we say that a smooth contour is *closed* if its initial and terminal points coincide, and a smooth closed contour is *simple* if it has no multiple points except for the terminal point of C_i being the initial point of C_{i+1}, for $1 \le i \le n-1$, and the terminal point of C_n being the inital point of C_1.

Jordan Curve Theorem. *A simple smooth closed contour Γ divides the complex plane into two open connected sets, an interior domain D_0 which is bounded, and an exterior domain D_1 which is unbounded.*

We call D_0 and D_1 the *connected components* of $\mathbb{C} - \Gamma$. The Jordan Curve Theorem is a relatively deep theorem from topology, and its proof is beyond the scope of the course.

4.2 Integration along contours

Most of the deepest results in complex analysis are proven most easily by means of integration. To obtain these results, we need to use some standard facts about integration which are described in calculus courses, and proven in real analysis courses such as Math 118. The standard text for 118 is Rudin [9], and Chapter 6 of this book gives proofs of the properties of the Riemann integral, the version of integral used in calculus.

Suppose that U is a connected open subset of $\mathbb{C}$ and $M(x,y)dx + N(x,y)dy$ is a differential on U which has continuously differentiable coefficients M and

N. If $z : [a,b] \to U$ is a parametrization of a smooth circected curve C within U, say

$$z(t) = x(t) + iy(t), \quad \text{for } a \le t \le b,$$

then the *contour integral* of $Mdx + Ndy$ along C is

$$\int_C Mdx + Ndy = \int_a^b \left[M(x(t), y(t)) \frac{dx}{dt} + N(x(t), y(t)) \frac{dy}{dt} \right] dt, \tag{4.2}$$

the right-hand side being a Riemann integral.

Example 1. Let $U = \mathbb{C} - \{0\}$ and consider the differential

$$M(x,y)dx + N(x,y)dy = \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$$

on U. Note that although M and N blow up at the origin, we have excised that point from U, so M and N are well-behaved, that is, have continuous first-order partial derivatives, on U. Suppose now that C is the counterclockwise unit circle $x^2 + y^2 = 1$. As parametrization of C we take

$$z : [0, 2\pi] \to \mathbb{C} \quad \text{defined by} \quad z(t) = e^{it} = \cos t + i \sin t,$$

so that

$$\begin{cases} x(t) = \cos t, \\ y(t) = \sin t, \end{cases} \qquad \text{for } 0 \le t \le 2\pi.$$

Then $x^2 + y^2 = 1$, so along C,

$$\begin{aligned} M(x,y)dx + N(x,y)dy &= -ydx + xdy \\ &= -(\sin t)d(\cos t) + \cos t d(\sin t) = \sin^2 t dt + \cos^2 t dt = dt, \end{aligned}$$

and (4.2) yields

$$\int_C Mdx + Ndy = \int_0^{2\pi} dt = 2\pi.$$

We claim that the contour integral (4.2) is invariant under reparametrization. To prove this, suppose that $\tau : [c,d] \to [a,b]$ is a continuously differentiable map with differentiable inverse, and note that

$$t = \tau(s) \quad \Rightarrow \quad dt = \frac{dt}{ds}(s) ds = \tau'(s) ds.$$

Thus it follows from the chain rule and the change of variable formula that

$$\begin{aligned}
&\int_c^d M(x\circ\tau(s), y\circ\tau(s))\frac{d(x\circ\tau)}{ds}ds \\
&\qquad = \int_c^d M(x\circ\tau(s), y\circ\tau(s))\frac{dx}{ds}(\tau(s))\tau'(s)ds \\
&\qquad = \int_c^d M(x\circ\tau(s), y\circ\tau(s))\frac{d(x\circ\tau)}{ds}(s)\frac{dt}{ds}(s)ds \\
&\qquad = \int_a^b M(x(t), y(t))\frac{dx}{dt}(t)dt. \qquad (4.3)
\end{aligned}$$

A similar argument shows that

$$\int_c^d N(x\circ\tau(s), y\circ\tau(s))\frac{d(y\circ\tau)}{ds}ds = \int_a^b N(x(t), y(t))\frac{dy}{dt}(t)dt. \qquad (4.4)$$

The sum of (4.3) and (4.4) is

$$\begin{aligned}
&\int_c^d M(x\circ\tau(s), y\circ\tau(s))\frac{d(x\circ\tau)}{ds}ds + \int_c^d N(x\circ\tau(s), y\circ\tau(s))\frac{d(y\circ\tau)}{ds}ds \\
&\qquad = \int_a^b M(x(t), y(t))\frac{dx}{dt}(t)dt + \int_a^b N(x(t), y(t))\frac{dy}{dt}(t)dt,
\end{aligned}$$

which implies that when τ is strictly increasing, so that $c < d$, the integral of $Mdx + Ndy$ along the directed curve C does not depend on the choice of parametrization.

On the other hand, when τ is strictly decreasing, so that $c > d$, we need to interchange the upper and lower limits, and this changes the sign, showing that

$$\int_{-C} Mdx + Ndy = -\int_C Mdx + Ndy.$$

More generally, suppose that

$$\Gamma = C_1 + C_2 + \cdots + C_n$$

is a smooth contour in the complex plane $\mathbb{C}$. Then the contour integral of the differential $Mdx + Ndy$ along Γ is

$$\int_\Gamma Mdx + Ndy = \int_{C_1} Mdx + Ndy + \cdots + \int_{C_n} Mdx + Ndy.$$

It follows that

$$\begin{aligned}
&\int_{-\Gamma} Mdx + Ndy = -\int_\Gamma Mdx + Ndy \quad \text{and} \\
&\qquad \int_{\Gamma_1+\Gamma_2} Mdx + Ndy = \int_{\Gamma_1} Mdx + Ndy + \int_{\Gamma_2} Mdx + Ndy.
\end{aligned}$$

Our goal is actually to calculate contour integrals of complex differentials. As we mentioned before, a *complex differential* on an open subset $U \subseteq \mathbb{C}$ is an expression of the form

$$f(z)dz, \quad \text{where} \quad f : U \to \mathbb{C}$$

is a continuous function. If we write f and dz in terms of real and imaginary parts,

$$f(x+iy) = u(x,y) + iv(x,y) \quad \text{and} \quad dz = dx + idy,$$

the differential can be rewritten as

$$\begin{aligned} f(z)dz &= (u(x,y) + iv(x,y))(dx + idy) \\ &= [u(x,y)dx - v(x,y)dy] + i\,[v(x,y)dx + u(x,y)dy]\,. \end{aligned}$$

Note that the real and imaginary parts of a complex differential are just ordinary differentials as described before. To integrate a complex differential along a smooth directed curve C, we choose a parametrization $z : [a,b] \to \mathbb{C}$ of C and integrate real and imaginary parts.

Indeed, if we write

$$z(t) = x(t) + iy(t), \quad dz = \frac{dz}{dt}dt = \left(\frac{dx}{dt} + i\frac{dy}{dt}\right)dt,$$

the *contour integral* of $f(z)dz$ along C is

$$\int_C f(z)dz = \int_a^b f(z)\frac{dz}{dt}(t)dt,$$

which is much more efficient to write than the expanded version,

$$\int_C f(z)dz = \int_a^b \left[u(x,y)\frac{dx}{dt} - v(x,y)\frac{dy}{dt}\right]dt + i\int_a^b \left[v(x,y)\frac{dx}{dt} + u(x,y)\frac{dy}{dt}\right]dt.$$

Example 2. Let $U = \mathbb{C} - \{z_0\}$, where z_0 is some point in $\mathbb{C}$ and let C be the circle of radius ρ about z_0 directed counterclockwise, which we can parametrize by (4.1). To calculate the contour integral

$$\int_C \frac{1}{z - z_0}dz \quad \text{of the differential} \quad \frac{1}{z-z_0}dz,$$

we write

$$z = z_0 + \rho e^{it}, \quad dz = i\rho e^{it}dt, \quad \frac{1}{z-z_0}dz = idt,$$

from which we can calculate

$$\int_C \frac{1}{z-z_0}dz = \int_0^{2\pi} idt = 2\pi i.$$

Note that the integral in this case is independent of the choice of the radius ρ.

If U is an open subset of $\mathbb{C}$ and $F : U \to \mathbb{C}$ is a complex analytic function, say

$$F(x+iy) = U(x,y) + iV(x,y),$$

then the *differential* of F is the complex differential

$$dF = F'(z)dz = \left(\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy\right) + i\left(\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy\right). \tag{4.5}$$

Note that we can rewrite (4.5) as

$$F'(z)dz = \left(\frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y}\right)(dx + idy), \quad \text{so} \quad F'(z) = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y}.$$

When the components of F have continuous second order partial derivatives, the fact that the real and imaginary parts of U are harmonic implies that $f'(z)$ satisfies the Cauchy-Riemann equations, so that the complex derivative of F is also complex analytic.

The fundamental theorem of calculus can be extended to say that the contour integral of a complex differential along a directed curve C is just the difference in the values of the function at the terminal and initial points of C.

Fundamental Theorem of Calculus for Contour Integrals. *If $F : U \to \mathbb{C}$ is a complex analytic function, then for any smooth arc C,*

$$\int_C F'(z)dz = F(z(b)) - F(z(a)), \quad \textit{whenever} \quad z : [a,b] \to \mathbb{C}$$

is a parametrization of C.

The idea behind the proof is to apply the chain rule and the usual fundamental theorem of calculus for Riemann integrals to the real and imaginary parts of F. Thus it follows from (4.5) that

$$\begin{aligned}\int_C F'(z)dz &= \int_a^b \left(\frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt}\right)dt + i\int_a^b \left(\frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dx}{dt}\right)dt \\ &= \int_a^b \frac{d}{dt}(U \circ z)dt + i\int_a^b \frac{d}{dt}(V \circ z)dt \\ &= U(z(b)) - U(z(a)) + iV(z(b)) - V(z(a)) = F(z(b)) - F(z(a)).\end{aligned}$$

Example 3. Let $U = \mathbb{C} - \{0\}$ and that $k \in \mathbb{Z}$ and $k \neq -1$. Then the function

$$F : U \to \mathbb{C} \quad \text{defined by} \quad F(z) = \frac{z^{k+1}}{k+1} \quad \text{has derivative} \quad F(z) = z^k.$$

Hence if C is any directed arc from z_0 to z_1,

$$\int_C z^k dz = \frac{z_1^{k+1}}{k+1} - \frac{z_0^{k+1}}{k+1}.$$

4.3 Independence of Path

The Fundamental Theorem of Calculus from the preceding section raises the following question: Suppose that U is a connected open subset of $\mathbb{C}$ and that $f(z)dz$ is a complex differential defined on U. We can then ask: Is there a complex analytic function $F : U \to \mathbb{C}$ such that

$$dF = F'(z)dz = f(z)dz?$$

In this case, we say that the function $f(z)$ has an *antiderivative* on U.

First, if $f(z)$ has an antiderivative, we would expect it to be complex analytic, because if $f = F'$ for some complex analytic function and the components of F have continuous derivatives up to order two, then F' is automatically complex analytic. Second, it follows from the Fundamental Theorem that if f has an antiderivative, then whenever C is a directed smooth closed curve in U parametrized by $z : [a, b] \to \mathbb{C}$, then

$$\gamma(a) = \gamma(b) \quad \Rightarrow \quad \int_C f(z)dz = \int_C F'(z)dz = F(\gamma(b)) - F(\gamma(a)) = 0,$$

that is, the contour integral of the differential around any closed curve within U is zero. The following Theorem shows that these two conditions are also sufficient:

Path Independence Theorem. *If U is a connected open subset of $\mathbb{C}$ and $f : U \to \mathbb{C}$ is a complex analytic function such that*

$$\int_\Gamma f(z)dz = 0 \tag{4.6}$$

whenever Γ is a closed contour within U, then f has an antiderivative; thus there is a complex analytic function $F : U \to \mathbb{C}$ such that $F'(z) = f(z)$.

The idea behind the proof is extremely simple. We choose a base point $z_0 = x_0 + iy_0$ within U. If $z_1 = x_1 + iy_1 \in U$, we then let

$$F(z_1) = \int_\Gamma f(z)dz,$$

when Γ is any smooth contour within U with initial point z_0 and terminal point z_1. This definition makes sense because if Γ_1 and Γ_2 are two smooth contours within U with initial point z_0 and terminal point z_1, then $\Gamma_2 - \Gamma_1$ is a closed contour and hence the hypothesis (4.6) implies that

$$\int_{\Gamma_2 - \Gamma_1} f(z)dz = 0.$$

But then

$$\int_{\Gamma_2} f(z)dz - \int_{\Gamma_1} f(z)dz = 0 \quad \text{or} \quad \int_{\Gamma_2} f(z)dz = \int_{\Gamma_1} f(z)dz.$$

To finish the proof of the Theorem, we need to show that F has a complex derivative at every point and that $F'(z_1) = f(z_1)$, for $z_1 \in U$. Since U is open, we can choose $\varepsilon > 0$ so that $N(z_1; \varepsilon) \subseteq U$. If we choose $\Delta z \in \mathbb{C} - \{0\}$ so that $|\Delta z| < \varepsilon$, then it follows from the definition of F that

$$F(z_1 + \Delta z) - F(z_1) = \int_\Gamma f(z)dz,$$

when Γ is any smooth contour with initial point z_1 and terminal point $z_1 + \Delta z$. If we choose Γ to be the usual constant speed straight line from z_1 to $z_1 + \Delta z$,

$$z(t) = z_1 + t\Delta z, \quad \text{for } 0 \leq t \leq 1,$$

then $dz = \Delta z dt$, and hence

$$F(z_1 + \Delta z) - F(z_1) = \int_0^1 f(z_1 + t\Delta z)\Delta z dt.$$

But then

$$\frac{F(z_1 + \Delta z) - F(z_1)}{\Delta z} = \int_0^1 f(z_1 + t\Delta z)dt,$$

and since f is continuous

$$\lim_{\Delta z \to 0} \frac{F(z_1 + \Delta z) - F(z_1)}{\Delta z} = f(z_1),$$

showing that the derivative $F'(z_1)$ exists and $F'(z_1) = f(z_1)$, which finishes the proof of the theorem.

Remark. The Path Independence Theorem is also true if we only assume (4.6) for closed polygonal contours.

4.4 Cauchy's Theorem

Definition. Suppose that U is a open subset of $\mathbb{C}$. We say that U is *simply connected* if it is connected and whenever simple Γ is a smooth closed contour within U, the bounded domain D_0 determined by Γ (via the Jordan Curve Theorem) is contained within U.

Thus if U is an open ball within $\mathbb{C}$ then U is simply connected, while if

$$U = \{z \in \mathbb{C} : z \neq 0\},$$

then U is not simply connected becaue the circle $|z| = 1$ lies within U while the open disk

$$D_0 = \{z \in \mathbb{C} : |z| < 1\}$$

it bounds is not contained in U.

Cauchy's Theorem. *Suppose that U is a subset of $\mathbb{C}$ which is open, connected and simply connected. If $f : U \to \mathbb{C}$ is a complex analytic function which has a continuous derivative $f' : U \to \mathbb{C}$, then*

$$\int_{\Gamma} f(z)dz = 0 \tag{4.7}$$

whenever Γ is a smooth closed contour within U.

Example 1. Suppose that we consider the differential

$$\frac{1}{z}dz.$$

It is not well-behaved at 0 so we start by setting $U = \mathbb{C} - \{0\}$. Then $f : U \to \mathbb{C}$ is complex analytic, but as we saw in Example 2 of §4.2, if C is the counterclockwise unit circle $|z| = 1$,

$$\int_{C} \frac{1}{z}dz = 2\pi i.$$

Thus the conclusion of Cauchy's Theorem does not hold, a reflection of the fact that U is not simply connected. But we can replace U by a smaller open set

$$U_1 = \mathbb{C} - \{ \text{ the negative } x\text{-axis } \},$$

which is simply connected. The unit circle is no longer contained in U_1, and the conclusion of Cauchy's Theorem holds. It now follows from the Path Independence Theorem that $f(z)$ has an antiderivative $F : U_1 \to \mathbb{C}$. Of course,

$$F(z) = \mathrm{Log}(z) + c,$$

the principal branch of the logarithm, up to addition of a complex constant c.

Example 2. Suppose next that we consider the differential

$$\frac{1}{\sqrt{1-z^2}}dz,$$

which is certainly badly behaved when $z = \pm 1$, but in addition is not well-defined until we pick a branch of the square root. We let

$$U = \mathbb{C} - ((-\infty, -1] \cup [1, \infty)),$$

and as in Example 2 of §3.3, we choose a branch by setting

$$\frac{1}{\sqrt{1-z^2}}dz = f(z)dz, \quad \text{where} \quad f(z) = \exp\left(-\frac{1}{2}\mathrm{Log}\left(1 - z^2\right)\right),$$

and Log is the principle value of the Logarithm. Then U is simply connected and $f : U \to \mathbb{C}$ is complex analytic, so the conclusion of Cauchy's Theorem holds and line integrals of $f(z)dz$ along smooth contours depend only on the endpoints. We can therefore define $F : U \to \mathbb{C}$ by

$$F(z_1) = \int_\Gamma f(z)dz,$$

where Γ is any smooth contour with initial point 0 and terminal point z_1. Then it follows from (3.18) that F can be thought of as the principal branch of multiple-valued function *arcsin* defined by

$$\text{Arcsin}(z_1) = \int_0^{z_1} \exp\left(-\frac{1}{2}\text{Log}\left(1 - z^2\right)\right) dz, \quad \text{for } z_1 \in U.$$

It is easier to define a branch of the arcsine via this formula than via (3.17),

$$\arcsin(z) = -i \log\left(iz \pm \sqrt{1 - z^2}\right),$$

which involves two multiple valued functions, the square root and the logarithm.

Proof of Cauchy's Theorem: Cauchy's Theorem actually follows directlly from Green's Theorem from vector calculus. Suppose that Γ is a simple closed contour within the simply connected open set U, let D_0 denote the bounded open connected subset of $\mathbb{C}$ determined by Γ, and let

$$\bar{D}_0 = D_0 \cup \Gamma.$$

We can think of Γ as the boundary of $\bar{D}_0$.

Green's Theorem. *Suppose that $M(x, y)dx + N(x, y)dy$ is a differential with coefficients M and N which have continuous partial derivatives on the region $\bar{D}_0$, and that Γ is directed counterclockwise. Then*

$$\int_\Gamma M dx + N dy = \int\int_{\bar{D}_0} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy.$$

An intuitive treatment of this theorem is presented in Math 5B (see for example §9.4 in Kreyszig [6]), while a more rigorous treatment is given in Math 118; In this course we will assume Green's Theorem as known.

We wish to apply Green's Theorem to a holomorphic differential $f(z)dz$ such that f has continuous derivative f' over U. Writing $f(z)$ and dz in terms of real and imaginary parts,

$$f(x + iy) = u(x, y) + iv(x, y) \quad \text{and} \quad dz = dx + idy,$$

gives us

$$f(z)dz = (u + iv)(dx + idy) = [udx - vdy] + i\,[vdx + udy]\,.$$

But it follows from Green's Theorem with $M = u$ and $N = -v$ that

$$\int_\Gamma udx - vdy = \int\int_{D_0} (-v_x - u_y)dxdy = 0. \tag{4.8}$$

Similarly, it follows from Green's Theorem with $M = v$ and $N = u$ and the other Cauchy-Riemann equation that

$$\int_\Gamma v dx + u dy = \int\int_{D_0} (u_x - v_y) dx dy = 0. \tag{4.9}$$

Then (4.8) and (4.9) yield

$$\int_\Gamma f(z) dz = 0,$$

finishing the proof of Cauchy's Theorem.

Cauchy's Theorem has the following important consequence:

Corollary. *Suppose that U is a subset of $\mathbb{C}$ which is open, connected and simply connected, and $f : U \to \mathbb{C}$ is a complex analytic function which has a continuous derivative $f' : U \to \mathbb{C}$, and that Γ_0 and Γ_1 are two contours within U which have the same initial and terminal points. Then*

$$\int_{\Gamma_0} f(z) dz = \int_{\Gamma_1} f(z) dz. \tag{4.10}$$

Indeed, we can set $\Gamma = \Gamma_0 - \Gamma_1$, and (4.10) is then an immediate consequence of (4.7).

4.5 Deformations of contours

Let U be a connected (but not necessarily simply connected) open subset of $\mathbb{C}$. Suppose that Γ_0 and Γ_1 are two smooth closed contours within U with piecewise smooth parametrization based on the same closed interval $[a, b]$:

$$z_0, z_0 : [a, b] \longrightarrow U.$$

We say that Γ_0 and Γ_1 are *homotopic* within U, or that Γ_1 is obtained from Γ_0 by deformation within U, if there is a continuous map

$$z : [0, 1] \times [a, b] \longrightarrow U$$

such that

$$z(0, t) = z_0(t), \quad z(1, t) = z_1(t), \quad \text{for } t \in [a, b],$$

and

$$z(s, a) = z(s, b), \quad \text{for } s \in [0, 1].$$

The *principle of deformation of contours* states that if $f : U \to \mathbb{C}$ is holomorphic, then

$$\int_{\Gamma_0} f(z) dz = \int_{\Gamma_1} f(z) dz,$$

whenever Γ_0 and Γ_1 are closed contours within U which can be obtained from each other via deformation within U. It is justified by the following theorem:

Homotopy Theorem. *Suppose that U is a connected (but not necessarily simply connected) open subset of $\mathbb{C}$, and $f : U \to \mathbb{C}$ is a complex analytic function which has a continuous derivative $f' : U \to \mathbb{C}$. If Γ_0 and Γ_1 are two closed contours within U which are homotopic within U, then*

$$\int_{\Gamma_0} f(z)dz = \int_{\Gamma_1} f(z)dz. \tag{4.11}$$

Here is a sketch of proof for the Homotopy Theorem in the case where z_0, z_1 and z have continuous partial derivatives up to order two. (Some readers may wish to skip the argument.) Note that a homotopy defines a one parameter family of contours Γ_s which are parametrized by

$$z_s : [a,b] \to \mathbb{C}, \quad z_s(t) = z(s,t), \quad \text{for } s \in [0,1]$$

It suffices to show that

$$\frac{d}{ds}\left[\int_{\Gamma_s} f(z)dz\right] = 0.$$

But our hypotheses imply that we can differentiate under the integral sign. Thus just in the argument presented for Theorem 8, page 186, in Saff and Snider [10], we can calculate this derivative, obtaining

$$\frac{d}{ds}\left[\int_{\Gamma_s} f(z)dz\right] = \frac{d}{ds}\left[\int_a^b (f\circ z_s)(t)\frac{dz_s}{dt}(t)dt\right]$$
$$= \int_a^b \left[(f'\circ z)\frac{\partial z}{\partial s}\frac{\partial z}{\partial t} + (f\circ z)\frac{\partial^2 z}{\partial s \partial t}\right](s,t)dt = \int_a^b \frac{d}{dt}\left((f\circ z)\frac{\partial z}{\partial s}\right)(s,t)dt$$
$$= (f\circ z)(s,b)\frac{\partial z}{\partial s}(s,b) - (f\circ z)(s,a)\frac{\partial z}{\partial s}(s,a) = 0.$$

We conclude that

$$\left[\int_{\Gamma_s} f(z)dz\right]$$

is constant as a function of s, which implies (4.11).

Remarks. Some authors derive versions of the Homotopy Theorem from Cauchy's Theorem (this is done in §49 of Brown and Churchill [2]), but it is also possible do derive Cauchy's Theorem from the Homotopy Theorem, as described in pages 180-191 of Saff and Snider [10]. To do this, one starts with an alternate definition of simply connected (which is shown to be equivalent to the other definition in topology courses).

Alternate Definition. An open subset U of $\mathbb{C}$ is said to be *simply connected* if it is connected and every closed contour Γ contained within U is homotopic to a trivial constant contour, parametrized by a smooth map to a point.

This implies that if U is simply connected and Γ_0 is any closed contour within U, then since Γ_0 is homotopic to a constant contour Γ_1 and the integral of any differential over a constant contour is zero,

$$\text{Homotopy Theorem} \quad \Rightarrow \quad \int_{\Gamma_0} f(z)dz = \int_{\Gamma_1} f(z)dz = 0,$$

whenever $f(z)dz$ is a holomorphic differential. The Homotopy Theorem is quite useful for calculating contour integrals, as the following examples illustrate.

Example 1. Suppose that we want to calculate the contour integral of the rational function

$$f(z) = \frac{3}{z-2} + \frac{7}{z-100},$$

around a counterclockwise circle Γ_0 of radius ten around the origin. To do this, we first note that f has poles at 2 and 100, but is holomorphic on the open set $U = \mathbb{C} - \{2, 100\}$. Thus if Γ_1 is any closed contour which is homotopic to Γ_0, the principle of deformation of contours states that

$$\int_{\Gamma_0} f(z)dz = \int_{\Gamma_1} f(z)dz.$$

We can choose Γ_1 to be a small circle of radius $\varepsilon > 0$, and let $V = N(2; 2\varepsilon)$. Then V is a simply connected open set containing Γ_1 and the function

$$\frac{7}{z-100}$$

is holomorphic on V, so it follows from Cauchy's theorem that

$$\int_{\Gamma_1} \frac{7}{z-100} dz = 0.$$

Thus

$$\begin{aligned} \int_{\Gamma_0} f(z)dz = \int_{\Gamma_1} \left(\frac{3}{z-2} + \frac{7}{z-100} \right) dz \\ = \int_{\Gamma_1} \frac{3}{z-2} dz = \int_0^{2\pi} 3e^{-it} ie^{it} dt = 6\pi i, \end{aligned}$$

where we have used the parametrization $z_1; [0, 2\pi] \to \mathbb{C}$ given by $z_1(t) = 2 + e^{it}$.

Example 2. Suppose that we want to calculate the contour integral of the rational function

$$f(z) = \frac{2}{z+1} + \frac{5}{z-1} + \frac{12}{z-100}.$$

along the counterclockwise circle Γ_0 of radius five centered at the origin, a contour which is homotopic to

$$\Gamma_1 = C_1 + C_2 + C_3 + C_4,$$

where C_1 is a counterclockwise circle of radius ε centered at -1 and beginning and ending at $-1+\varepsilon$, C_2 is the line segment from $-1+\varepsilon$ to $1-\varepsilon$, C_3 is the couterclockwise circle of radius ε centered at 1 starting and ending at $1-\epsilon$, and $C_4 = -C_2$. Thus

$$\int_{\Gamma_0} f(z)dz = \int_{\Gamma_1} f(z)dz = \int_{C_1} f(z)dz + \int_{C_3} f(z)dz,$$

since the contour integrals over C_2 and C_4 cancel. Using the same procedure as in Example 1, we find that

$$\int_{C_1} f(z)dz = \frac{2}{z+1}dz = 4\pi i$$

and

$$\int_{C_3} f(z)dz = \frac{5}{z-1}dz = 10\pi i,$$

so

$$\int_{\Gamma_0} \left(\frac{2}{z+1} + \frac{5}{z-1} + \frac{12}{z-100} \right) dz = 14\pi i.$$

4.6 Improper integrals of rational functions

The examples at the end of the previous section illustrate how Cauchy's Theorem and the Principle of Deformation of Contours can be used to calculate contour integrals of any rational function (as studied in §3.1) which has been expanded in a partial fractions decomposition. Suppose, for example that

$$R(z) = \frac{A_1}{z-\zeta_1} + \frac{A_2}{z-\zeta_2} + \cdots + \frac{A_n}{z-\zeta_n},$$

where $\zeta_a, \ldots, \zeta_n$ are the poles of $R(z)$. In this case we call A_i the *residue* at ζ_i. If Γ is a counterclockwise simple closed contour which misses all of the poles, we find that the contour integral of $R(z)$ around Γ is $2\pi i$ times the sum of all of the poles that are enclosed by Γ. Thus

$$\int_{\Gamma} R(z)dz = 2\pi i \left(\sum_{\zeta_i \in D_0} A_i \right),$$

where D_0 is the bounded open connected component of $\mathbb{C} - \Gamma$; thus ζ_i *is enclosed by* Γ when $\zeta_i \in D_0$.

Example 1. Suppose that

$$R(z) = \frac{3}{z-1} + \frac{6}{z} + \frac{2}{z+2} + \frac{55}{z-100} + \frac{67}{z-200},$$

and Γ is the counterclockwise circle of radius 5 centered at the origin. Then the poles

$$1, 0, -2 \in D_0 = \{z \in \mathbb{C} : |z| < 5\},$$

so

$$\int_\Gamma R(z)dz = 2\pi i(3+6+2) = 22\pi i.$$

This technique has important applications. Thus suppose that

$$R(z) = \frac{P(z)}{Q(z)},$$

where

$$P(z) = a_m z^m + a_{m-1}z^{m-1} + \cdots + a_1 z + a_0$$
$$\text{and} \quad Q(z) = b_n z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0$$

are polynomials with real coefficients

$$a_m,\ a_{m-1},\ \ldots,\ a_1,\ a_0, \quad b_n,\ b_{n-1},\ \ldots,\ b_1,\ b_0,$$

and we make the assumptions that

1. $Q(z)$ has no real roots, and
2. $\deg Q \geq \deg P + 2$.

It is then possible to use the *method of contour integrals* to evaluate the improper integral

$$\int_{-\infty}^{\infty} R(x)dx = \lim_{a\to\infty}\int_{-a}^{a} R(x)dx,$$

an integral which may be difficult or impossible to evaluate by the techniques of standard calculus. We illustrate how this is done with an example:

Example 2. Suppose that we want to calculate the integral

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}dx = \lim_{a\to\infty}\int_{-a}^{a} \frac{1}{(x^2+1)^2}dx.$$

To solve this problem, we first extend the integrand to be a complex differential

$$\frac{1}{(z^2+1)^2}dz,$$

and then instead of just integrating over real intervals, we can instead take contour integrals over any contour in the complex plane.

In order to take contour integrals, we need a partial fraction decomposition of the rational function

$$R(z) = \frac{1}{(z^2+1)^2} = \frac{1}{(z+i)^2(z-i)^2},$$

so we write

$$R(z) = \frac{A_{1,0}}{(z-i)^2} + \frac{A_{1,1}}{(z-i)} + \frac{A_{2,0}}{(z+i)^2} + \frac{A_{2,1}}{(z+i)}$$

where, according to (3.6),

$$A_{1,0} = \lim_{z\to i}(z-i)^2 R(z) = \lim_{z\to i} \frac{1}{(z+i)^2} = \frac{1}{(2i)^2} = -\frac{1}{4},$$

$$\begin{aligned} A_{1,1} = \lim_{z\to i} \frac{d}{dz}\left((z-i)^2 R(z)\right) \\ = \lim_{z\to i} \frac{d}{dz}\left(\frac{1}{(z+i)^2}\right) = \lim_{z\to i}\left(\frac{-2}{(z+i)^3}\right) = \frac{-2}{(2i)^3} = -\frac{i}{4}, \end{aligned}$$

and by conjugation, we obtain

$$A_{2,0} = -\frac{1}{4}, \quad A_{2,1} = \frac{i}{4}.$$

Thus

$$R(z) = \frac{1}{(z^2+1)^2} = \frac{1/4}{(z-i)^2} + \frac{-i/4}{z-i} + \frac{1/4}{(z+i)^2} + \frac{i/4}{z+i}. \tag{4.12}$$

Note that $R(z)$ is well-behaved except at the two poles $z = i$ and $z = -i$, and if

$$U = \mathbb{C} - \{i, -i\}, \quad \text{then} \quad R : U \to \mathbb{C}$$

is holomorphic.

Suppose now that Γ is a counterclockwise circle of small radius $\varepsilon > 0$ centered at the pole i. Then

$$\int_\Gamma \left[\frac{1/4}{(z+i)^2} + \frac{i/4}{z+i}\right] dz = 0,$$

by Cauchy's Theorem, because the expression within brackets is holomorphic on a small simply connected ball centered at i which contains Γ. Moreover,

$$\frac{1/4}{(z-i)^2} = \frac{d}{dz}\left(\frac{-1/4}{z-i}\right),$$

so

$$\int_\Gamma \frac{1/4}{(z-i)^2} dz = 0.$$

Thus

$$\int_\Gamma R(z)dz = \frac{-i}{4}\int_\Gamma \frac{1}{z-i}dz = \left(\frac{-i}{4}\right)(2\pi i) = \frac{\pi}{2}.$$

By the principle of deformation of contours if Γ is *any* contour in U homotopic within U to the counterclockwise circle of radius ε about i, then

$$\int_\Gamma \frac{1}{(z^2+1)^2} dz = \frac{\pi}{2}.$$

We now take $\Gamma = C_1 + C_2$, where C_1 is the directed line segment from $-a$ to a along the x-axis, and C_2 is the counterclockwise semicircle parametrized by

$$z(t) = ae^{it}, \quad \text{for } 0 \leq t \leq \pi.$$

Then

$$\int_{C_1} \frac{1}{(z^2+1)^2} dz + \int_{C_2} \frac{1}{(z^2+1)^2} dz = \frac{\pi}{2},$$

or

$$\int_{-a}^{a} \frac{1}{(x^2+1)^2} dx + \int_0^{\pi} \frac{1}{((ae^{it})^2+1)^2} iae^{it} dt = \frac{\pi}{2}.$$

But

$$\left| \int_0^{\pi} \frac{1}{((ae^{it})^2+1)^2} iae^{it} dt \right| \leq \int_0^{\pi} \frac{a}{(a^2-1)^2} dt = \frac{\pi a}{(a^2-1)^2},$$

so

$$\lim_{a \to \infty} \int_0^{\pi} \frac{1}{((ae^{it})^2+1)^2} iae^{it} dt = 0$$

and

$$\lim_{a \to \infty} \int_{-a}^{a} \frac{1}{(x^2+1)^2} dx = \frac{\pi}{2} \quad \text{or} \quad \int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2} dx = \frac{\pi}{2}.$$

4.7 The Cauchy integral formulae

One of the central results of complex analysis is the Cauchy Integral Theorem, which implies that the values of a complex analytic function defined on a closed disk are completely determined by the values on the boundary of the disk:

Cauchy Integral Formula I. *Suppose that U is a simply connected open subset of $\mathbb{C}$, and that Γ is a counterclockwise simple closed contour within U such that the point z_0 lies inside Γ. Then if $f : U \to \mathbb{C}$ is a holomorphic function,*

$$f(z_0) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z - z_0} dz. \tag{4.13}$$

To prove this, we note first that the integrand

$$\frac{f(z)}{z - z_0}$$

is a holomorphic function of z on $U - \{z_0\}$. Thus by the Homotopy Theorem,

$$\int_{\Gamma} \frac{f(z)}{z - z_0} dz = \int_{C_\varepsilon} \frac{f(z)}{z - z_0} dz,$$

where C_ε is the circle of radius ε centered at z_0. Of course, we have a standard parametrization

$$z : [0, 2\pi] \to U, \quad z(t) = z_0 + \varepsilon e^{it},$$

and using this parametrization yields

$$\int_{C_\delta} \frac{f(z)}{z-z_0} dz = \int_0^{2\pi} \frac{f(z_0+\varepsilon e^{it})}{\varepsilon e^{it}} i\varepsilon e^{it} dt = \int_0^{2\pi} f(z_0+\varepsilon e^{it}) i dt,$$

so

$$\int_\Gamma \frac{f(z)}{z-z_0} dz = \lim_{\varepsilon\to 0} \int_{C_\varepsilon} \frac{f(z)}{z-z_0} dz = \lim_{\varepsilon\to 0} \int_0^{2\pi} f(z_0+\varepsilon e^{it}) i dt = 2\pi i f(z_0),$$

which implies (4.13).

Note that we can rewrite (4.13) as

$$\int_\Gamma \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0). \tag{4.14}$$

Example 1. Suppose that we want to calculate the contour integral

$$\int_\Gamma \frac{e^z}{z-i} dz,$$

where Γ is a small counterclockwise circle which encloses the singularity at $z = i$. To carry this out, we can apply (4.14) with $f(z) = e^z$ to obtain

$$\int_\Gamma \frac{e^z}{z-i} dz = 2\pi i f(i) = 2\pi i e^i = (2\pi i)(\cos 1 + i \sin 1)$$

by Euler's formula. Here angles are measured in radians, one radian is $180/\pi$ degrees, and use of a calculator shows that

$$\cos 1 = .540302, \quad \sin 1 = .841471.$$

Example 2. Suppose that we want to calculate the contour integral

$$\int_{\Gamma_0} \frac{2e^z}{z^2-1} dz,$$

where Γ_0 is the counterclockwise circle of radius five about the origin. As a first step, we need to construct a partial fraction expansion

$$\frac{2}{z^2-1} = \frac{A}{z+1} + \frac{B}{z-1}.$$

This is relatively easy and the result is

$$\frac{2}{z^2-1} = \frac{-1}{z+1} + \frac{1}{z-1},$$

so

$$\int_{\Gamma_0} \frac{2e^z}{z^2-1} dz = \int_{\Gamma_0} \frac{-e^z}{z+1} dz + \int_{\Gamma_0} \frac{e^z}{z-1} dz.$$

Next we use the Homotopy Theorem as in Example 2 of §4.5 to replace the contour integral over Γ_0 by a sum of contour integrals over counterclockwise circles C_1 and C_3 of radius ε centered at -1 and 1 respectively. Thus

$$\int_{\Gamma_0} \frac{2e^z}{z^2-1} dz = \int_{C_1} \frac{-e^z}{z+1} dz + \int_{C_3} \frac{e^z}{z-1} dz.$$

Finally, we apply (4.14) with $f(z) = e^z$ to obtain

$$\int_{C_1} \frac{-e^z}{z+1} dz = -2\pi i f(-1) = -2\pi i e^{-1} = -(2\pi i)e^{-1},$$

$$\int_{C_3} \frac{e^z}{z-1} dz = 2\pi i f(1) = 2\pi i e^1 = (2\pi i)e,$$

which leads to the final answer

$$\int_{\Gamma_0} \frac{2e^z}{z^2-1} dz = 2\pi i(e - e^{-1}) = 4\pi i \sinh(1).$$

We can differentiate the Cauchy integral formula (4.13) with respect to z_0 to obtain a formula for higher derivatives of a holomorphic function f:

Cauchy Integral Formula II. *Suppose that U is a simply connected open subset of $\mathbb{C}$, and that Γ is a counterclockwise simple closed contour within U such that the point z_0 lies inside Γ. Then if $f : U \to \mathbb{C}$ is a holomorphic function,*

$$\frac{d^k f}{dz^k}(z_0) = \frac{k!}{2\pi i} \int_\Gamma \frac{f(z)}{(z-z_0)^{k+1}} dz. \tag{4.15}$$

We can of course rewrite this formula as

$$\int_\Gamma \frac{f(z)}{(z-z_0)^{k+1}} dz = \frac{2\pi i}{k!} \frac{d^k f}{dz^k}(z_0). \tag{4.16}$$

Example 3. Suppose that we want to calculate the contour integral

$$\int_\Gamma \frac{e^z}{(z+1)^5} dz,$$

where Γ is a small counterclockwise circle around the pole at $z = -1$. In this case, we can apply (4.16) in the case where $f(z) = e^z$ and since

$$\frac{d^k f}{dz^k}(z) = e^z,$$

we obtain

$$\int_\Gamma \frac{e^z}{(z+1)^5} dz = \frac{2\pi i}{4!} e^{-1} = \frac{\pi i}{12e}.$$

Theoretical implications of the Cauchy integral formulae: The Cauchy Integral Theorem II implies that a complex function $f : U \to \mathbb{C}$ which has a continuous complex derivative at every point of U has continuous derivatives of all orders. Indeed, one can go further. In 1900, Édouard Goursat was able to prove Cauchy's Theorem without the redundant assumption that f' be continuous. He merely assumed that $f : U \to \mathbb{C}$ has a complex derivative at each point of U. (Goursat's proof for rectangular regions is presented at the beginning of Chapter 4 of Ahlfors [1].) Since one can prove Cauchy's Integral Theorem I using Cauchy's Theorem, it follows that merely assuming that $f : U \to \mathbb{C}$ has a complex derivative at each point of U implies that f has a continuous complex derivative $f' : U \to \mathbb{C}$, and then via the Cauchy Integral Theorem II that f has continuous complex derivatives of all orders. Needless to say, these remarkable facts do not hold for ordinary derivatives of real-valued functions of one real variable.

4.8 Applications of Cauchy's integral formulae

We now have enough machinery to use complex analysis to prove some striking results from other branches of mathematics, that any nonconstant polynomial with complex coefficients must have at least one complex root and that solutions to Laplace's equations cannot have maxima (ore minima) except at boundary points.

Cauchy Estimate Theorem. *Suppose that f is complex analytic on*

$$\bar{D}_R(z_0) = \{z \in \mathbb{C} : |z - z_0| \leq R\}.$$

If $|f(z)| \leq M$ for all

$$z \in \partial \bar{D}_R(z_0) = \{z \in \mathbb{C} : |z - z_0| = R\},$$

then

$$\left| \frac{d^k f}{dz^k}(z_0) \right| \leq \frac{k! M}{R^k}. \tag{4.17}$$

This follows from the Cauchy Integral Theorem II. If we use the standard parametrization

$$z : [0, 2\pi] \to \mathbb{C}, \quad z(t) = z_0 + Re^{it}$$

of the counterclockwise circle of radius R about z_0, then (4.15) implies

$$\left| \frac{d^k f}{dz^k}(z_0) \right| \leq \int_0^{2\pi} \frac{M}{R^{k+1}} R dt = \frac{k! M}{R^k}.$$

Liouville's Theorem. *Any bounded entire function is constant.*

Indeed, if $f : \mathbb{C} \to \mathbb{C}$ is a bounded complex analytic function, say $|f(z)| \leq M$ for all $z \in \mathbb{C}$, and $z_0 \in \mathbb{C}$, then (4.17) implies that

$$\left| \frac{d^k f}{dz^k}(z_0) \right| \leq \frac{k!M}{R^k} \quad \text{for all } R > 0,$$

which implies that

$$\frac{d^k f}{dz^k}(z_0) = 0, \quad \text{for all } k > 0,$$

and hence f must be constant.

Fundamental Theorem of Algebra. *Any nonconstant polynomial with complex coefficients has at least one complex root.*

Proof: Suppose that

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

is a polynomial with complex coefficients with $a_n \neq 0$ which has no roots, where $n \geq 1$. Then there exists $R > 0$ such that

$$P(z) \geq \frac{|a_n|}{2} |z|^n \geq \frac{|a_n|}{2} R^n, \quad \text{for} \quad |z| > R,$$

where we can assume that $R \geq 1$, so that

$$\frac{1}{P(z)} \leq \frac{2}{|a_n|} \frac{1}{R^n} \leq \frac{2}{|a_n|}, \quad \text{for} \quad |z| > R.$$

On the other hand, the modulus of P,

$$|P| : \bar{D}_R \to \mathbb{R} \quad \text{where} \quad \bar{D}_R = \{z \in \mathbb{C} : |z| \leq R\},$$

is continuous and positive, so it must assume its greatest lower bound $\varepsilon_0 > 0$ at some point of $\bar{D}_R$, by one of the standard theorems of real analysis. Thus

$$\frac{1}{|P(z)|} \leq \begin{cases} 2/|a_n|, & \text{when } |z| > R, \\ 1/\varepsilon_0, & \text{when } |z| \leq R. \end{cases}$$

Hence $1/P(z)$ is a bounded entire function, and it must therefore be constant by Liouville's Theorem. QED

Another implication of the Cauchy Integral Theorem is the fact that if $f : D \to \mathbb{C}$ is a complex analytic function on a bounded open connected subset $D \subseteq \mathbb{C}$, its modulus $|f|$ must achieve its maximum value on the boundary of D

Maximum Modulus Principle. *Suppose that D is a simply connected open set, that $f : D \to \mathbb{C}$ is complex analytic and $|f|$ achieves its maximum value at some point $z_0 \in D$, then f is constant within D.*

Idea of proof: By the Cauchy Integral Theorem (4.13), if Γ is the circle of radius R about z_0 parametrized by $z(t) = z_0 + Re^{it}$, for $t \in [0, 2\pi]$, then

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z - z_0} dz \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + Re^{it})}{Re^{it}} iRe^{it} dt = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{it}) dt. \end{aligned} \tag{4.18}$$

This equation says that $f(z_0)$ is the average of f around any circle $\Gamma_R(z_0)$ of radius R centered at z_0 which is contained within D. It implies an inequality on the modulus

$$|f|(z_0) \le \frac{1}{2\pi}\int_0^{2\pi} |f|(z_0 + Re^{it}) dt. \tag{4.19}$$

If z lies on $\Gamma_R(z_0)$ then $|f|(z) \le |f|(z_0)$, but if $|f|(z) < |f|(z_0)$, then the average value of $|f|$ on the circle would be strictly less that $|f|(z_0)$, contradicting (4.19). We conclude that $|f|$ must be constant on D.

Thus $\mathrm{Re}(e^f) = e^{|f|}$ is constant on D. It then follows from the Cauchy-Riemann equations (Proposition 2 from §2.3) that e^f is constant. But then

$$0 = \frac{d}{dz}(e^f) = e^f \frac{df}{dz} \quad \Rightarrow \quad \frac{df}{dz} = 0,$$

by the chain rule, so f itself is constant on D.

Maximum Principle for Harmonic Functions. *Suppose that D is a simply connected open subset of $\mathbb{C}$ and that $u : D \to \mathbb{R}$ is a harmonic function. If u achieves its maximum value at some point $z_0 \in D$, then u is constant on D.*

Sketch of proof: The fact that D is simply connected allows us to construct an harmonic conjugate v to u. Indeed, using the fact that u is harmonic, one verifies that if

$$M = \frac{\partial u}{\partial x} \quad \text{and} \quad N = \frac{\partial u}{\partial y}$$

then $M - iN$ is holomorphic, and hence

$$(M - iN)(dx + idy) = f(z)dz$$

is a holomorphic differential on D such that $\mathrm{Re}(f(z)dz) = du$. Since D is simply connected, it follows from Cauchy's Theorem that whenever Γ is a closed contour within D,

$$\int_\Gamma f(z)dz = 0.$$

Then it follows from the Independence of Path Theorem from §4.3 that $f(z)$ has an antiderivative F on D. In other words, there is a holomorphic function $F : D \to \mathbb{C}$ such that $dF = f(z)dz$, and hence

$$\mathrm{Re}(dF) = \mathrm{Re}(f(z)dz) = du.$$

Finally, we set $v = \mathrm{Im}(F)$, and $F = u + iv$, so v is an harmonic conjugate to u.

Then e^F is also a complex analytic function with

$$\left|e^F\right| = \left|e^{u+iv}\right| = e^u.$$

If u assumes its maximum at some point $z_0 \in D$, then e^F achieves its maximum modulus at $z_0 \in D$ and hence e^F is constant on D by the maximum modulus principle. But then e^u and hence u itself must be constant on D.

The Maximum Principle can be applied in turn to the so-called Dirichlet Problem for Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

solutions to which are called harmonic functions. Recall that if D is an open connected subset of the complex plane $\mathbb{C}$ filled with a homogeneous material, then a harmonic function $u : D \to \mathbb{R}$ might represent the steady-state temperature within D, as we saw in §3.4.

Dirichlet Problem. *Suppose that D is the closure of a bounded open connected set in $\mathbb{C}$ with boundary ∂D. Given the temperature along the boundary ∂D of D, determine the temperature within D; that is, given $u|\partial D$, determine the function $u : D \to \mathbb{R}$ which satisfies Laplace's equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

within D and extends continuously to $D \cup \partial D$.

The physical interpretation in which harmonic functions represent steady-state temperature suggests that there should exist a unique solution to Dirichlet's problem, at least if D is reasonably well-behaved. The maximum principle shows that this is indeed true:

Uniqueness of solutions to the Dirichlet Problem. *Suppose that D is the closure of a bounded open connected set in $\mathbb{C}$ with boundary ∂D. If u and v are two solutions to the Dirichlet problem for D with the same values on ∂D, then $u = v$.*

Indeed, $u - v$ is a solution to the Dirichlet Problem with zero boundary conditions. By the Maximum Principle applied to $u - v$, $u - v \leq 0$ on D. By the Maximum Principle applied to $v - u$, $v - u \leq 0$ on D. Hence $u - v = 0$ on D.

Poisson's formula: Yet another application of the Cauchy integral formula is Poisson's explicit formula for the solution to the Dirichlet problem in the unit disk

$$D = \{z \in \mathbb{C} : |z| < 1\}.$$

We derive this formula following Ahlfors [1], pages 165-166, noting first that it follows from (4.18) that if f is a holomorphic function on the disk and Γ is the boundary of the disk,

$$f(0) = \frac{1}{2\pi}\int_\Gamma f(z)\frac{dz}{iz}. \tag{4.20}$$

For a fixed choice of $a \in D$, we now make a change of variables, using a linear fractional transformation, as described in §3.2:

$$w = T(z) = \frac{z-a}{1-\bar{a}z}, \quad \text{so} \quad T(a) = 0.$$

This linear fractional transformation takes the unit disk D to itself. It follows immediately that

$$\frac{dw}{w} = \frac{dz}{z-a} + \frac{\bar{z}dz}{1-\bar{a}z}.$$

We now replace f by the holomorphic function $f \circ T^{-1}$ in (4.20), thereby obtaining

$$\begin{aligned} f(a) = f(T^{-1}(0)) &= \frac{1}{2\pi}\int_\Gamma f(T^{-1}(w))\frac{dw}{iw} \\ &= \frac{1}{2\pi}\int_\Gamma f(z)\left(\frac{z}{z-a} + \frac{\bar{a}z}{1-\bar{a}z}\right)\frac{dz}{iz} \\ &= \frac{1}{2\pi}\int_\Gamma f(z)\left(\frac{z}{z-a} + \frac{\bar{a}}{\bar{z}-\bar{a}}\right)\frac{dz}{iz} = \frac{1}{2\pi}\int_\Gamma f(z)\frac{1-|a|^2}{|z-a|^2}\frac{dz}{iz}, \end{aligned}$$

where we have used the fact that $|z| = 1$. If we use the parametrization $z(t) = e^{it}$, for $0 \le t \le 2\pi$, we can write this as

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\frac{1-|a|^2}{|e^{it}-a|^2}dt. \tag{4.21}$$

If u is a harmonic function on the disk, it is the real part of a holomorphic function of f, and hence it follows from (4.21) that

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{it})\frac{1-|a|^2}{|e^{it}-a|^2}dt,$$

an explicit formula for the value of the harmonic function u at $a \in D$ in terms of the values of u along Γ.

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