# Elementary Row Operations

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John Douglas Moore

**A. Homogeneous linear systems**

Linear algebra is the theory behind solving systems of linear equations, such as

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & 0, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & 0, \\ \cdots\cdots & \cdot & \cdot \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & 0. \end{array} \tag{1}$$

Here the $a_{ij}$'s are known elements of the field $\mathbb{F}$, and we are solving for the unknown elements $x_1, \ldots, x_n$ in $\mathbb{F}$. Our goal is to describe the space of solutions

$$W = \{(x_1, x_2, \ldots, x_n) \in \mathbb{F}^n : \ (x_1, x_2, \ldots, x_n) \text{ satisfies (1) } \}$$

as simply as possible. You will note that $W$ is a linear subspace of $\mathbb{F}^n$. In fact, it turns out that any linear subspace of $\mathbb{F}^n$ is a solution set to a homogeneous linear system of equations just like (1).

Our strategy is to simplify the system by means of the *elementary operations on equations*:

1. Interchange two equations.
2. Multiply an equation by a nonzero constant $c$.
3. Add a constant multiple of one equation to another.

All of these operations are reversible and each leads to a new system with exactly the same solution set $W$. We want to choose these operations judiciously, so that we put the system into the simplest possible form.

From properties of matrix multiplication that you learned in Math 3C, you realize that this system of linear equations can be written in terms of its *coefficient matrix*

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad \text{and the vector} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ x_n \end{pmatrix}.$$

as $A\mathbf{x} = \mathbf{0}$. Thus the solution set $W$ can be expressed more simply as

$$W = \{\mathbf{x} \in \mathbb{F}^n : A\mathbf{x} = \mathbf{0}\}.$$

Each elementary operation on the linear system (1) corresponds to elementary row operations on its coefficient matrix $A$. Those *elementary row operations* are:

1. Interchange two rows.
2. Multiply a row by a nonzero constant $c$.
3. Add a constant multiple of one row to another.

Each of these operations is reversible and leaves the solutions to the matrix equation $A\mathbf{x} = \mathbf{0}$ unchanged. Our goal is to use these operations to replace $A$ by a matrix that is in row-reduced echelon form. By definition, the matrix $A$ is in *row-reduced echelon form* if it has the following properties:

1. The first nonzero entry in any row is a one.
2. If a column contains an initial one for some row all of the other entries in that column are zero.
3. The initial one in a given row occurs to the right of the initial ones in all higher rows.
4. If a row consists of all zeros, then it is below all of the other rows.

It can be proven that any $m \times n$ matrix can be put in row-reduced echelon form by elementary row operations, and the reduced row-reduced echelon matrix is unique. It is easy to see how to carry out the procedure. One starts by putting a one as the first nonzero entry in the first row of the first nonzero column. We do this by interchanging rows if necessary to get a nonzero entry in the first nonzero column into the first row, and then divide the first row by this first nonzero entry. We then zero out all other elements in the first nonzero column by subtracting suitable multiples of the first row from other rows. In the submatrix obtained by removing the first row, we then apply the same procedure obtaining an initial one in a second column and zeroing out all other entries in that column. Continuing with this procedure leads to a matrix in row-reduced echelon form.

Indeed, properly reformulated, the procedure we have described could be refined to give a proof that any matrix can be put in row-reduced echelon form by elementary row operations. We will not carry out all of the details, but you should be able to see how they would go.

If there are $k$ nonzero rows, one can solve for the $k$ variables corresponding to the initial ones in those rows in terms of the $n-k$ variables corresponding to the other rows. The $n-k$ variables which do NOT correspond to initial ones are called *free variables*, and can be thought of as coordinates for the space $W$. In this way, we can obtain a general solution to the original linear system, in which the $n-k$ free variables form the coordinates.

**Example 1.** Suppose we consider the homogeneous linear system

$$\begin{array}{ccccccccc} 2x_1 & + & 4x_2 & + & 2x_3 & + & 4x_4 & = & 0, \\ & & x_2 & + & x_3 & & & = & 0, \\ x_1 & + & 3x_2 & + & 2x_3 & + & 2x_4 & = & 0. \end{array} \tag{2}$$

We write out the coefficient matrix, and apply the elementary row operations:

$$\begin{pmatrix} 2 & 4 & 2 & 4 \\ 0 & 1 & 1 & 0 \\ 1 & 3 & 2 & 2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 1 & 3 & 2 & 2 \end{pmatrix}$$

$$\mapsto \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mapsto \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The last matrix is in row-reduced echelon form. Our linear system (2) has the same solution set as the system

$$\begin{array}{ccccccccc} x_1 & & & - & x_3 & + & 2x_4 & = & 0, \\ & & x_2 & + & x_3 & & & = & 0, \\ & & & & & & 0 & = & 0. \end{array} \tag{3}$$

We can rewrite (2) in the form

$$\begin{array}{cccc} x_1 & = & x_3 & -2x_4, \\ x_2 & = & -x_3 & , \\ x_3 & = & x_3 & , \\ x_4 & = & & x_4. \end{array}$$

The free variables in the system are $x_3$ and $x_4$; we can set them to any elements of $\mathbb{F}$ and then $x_1$ and $x_2$ are completely determined. Replacing $x_3$ by $s$ and $x_4$ by $t$, we can rewrite this system as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = s \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

where the parameters $s$ and $t$ range over $\mathbb{F}$. Since

$$W = \left\{ s \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} : s, t \in \mathbb{F} \right\}$$

we say that the vectors

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{b}_2 = \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

*span* the vector space $W$. Since

$$s \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 0 \quad \Rightarrow \quad s = t = 0,$$

we say that the list of vectors $(\mathbf{b}_1, \mathbf{b}_2)$ is *linearly independent*. Since the list of vectors $(\mathbf{b}_1, \mathbf{b}_2)$ spans $W$ and is linearly independent, we say it is a *basis* for the linear subspace $W$.

To summarize, the elementary row operations allow one to find a basis for the vector space $W$ of solutions to the homogeneous linear system (1). Although the basis for $W$ is not unique (other bases could be constructed by other methods), it can be proven that all bases have the same number of elements. The number of elements in a basis is called the *dimension* of the subspace $W$. Bases for a solution set $W$ to a homogeneous linear system provide the sought-after simplest possible descriptions of the solution set.

## B. Nonhomogeneous linear systems

We can use a similar procedure to find the general solution to nonhomogeneous linear systems, such as

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2, \\ \cdots\cdots & \cdot & \cdot \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_2. \end{array} \tag{4}$$

In this case, the $a_{ij}$'s and the $b_i$'s are known elements of the field $\mathbb{F}$, and we are solving for the unknown elements $x_1, \ldots, x_n$ in $\mathbb{F}$. We can write this system as $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ x_n \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ b_m \end{pmatrix}.$$

To solve such a system, we apply the elementary row operations to the *augmented coefficient matrix*

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & | & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & | & b_2 \\ \cdot & \cdot & \cdots & \cdot & | & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} & | & b_m \end{pmatrix}.$$

**Example 2.** Let us consider the nonhomogeneous linear system

$$\begin{array}{ccccccccc} 2x_1 & + & 4x_2 & + & 2x_3 & + & 4x_4 & = & 8, \\ & & x_2 & + & x_3 & & & = & 3, \\ x_1 & + & 3x_2 & + & 2x_3 & + & 2x_4 & = & 7. \end{array} \tag{5}$$

We write our the augmented coefficient matrix and apply the elementary row operations:

$$\left(\begin{array}{cccccc} 2 & 4 & 2 & 4 & | & 8 \\ 0 & 1 & 1 & 0 & | & 3 \\ 1 & 3 & 2 & 2 & | & 7 \end{array}\right) \mapsto \left(\begin{array}{cccccc} 1 & 2 & 1 & 2 & | & 4 \\ 0 & 1 & 1 & 0 & | & 3 \\ 1 & 3 & 2 & 2 & | & 7 \end{array}\right)$$

$$\mapsto \left(\begin{array}{cccccc} 1 & 2 & 1 & 2 & | & 4 \\ 0 & 1 & 1 & 0 & | & 3 \\ 0 & 1 & 1 & 0 & | & 3 \end{array}\right) \mapsto \left(\begin{array}{cccccc} 1 & 2 & 1 & 2 & | & 4 \\ 0 & 1 & 1 & 0 & | & 3 \\ 0 & 0 & 0 & 0 & | & 0 \end{array}\right)$$

$$\mapsto \left(\begin{array}{cccccc} 1 & 0 & -1 & 2 & | & -2 \\ 0 & 1 & 1 & 0 & | & 3 \\ 0 & 0 & 0 & 0 & | & 0 \end{array}\right).$$

In this case, our linear system (5) has the same solution set as the system

$$\begin{array}{ccccccccc} x_1 & & & - & x_3 & + & 2x_4 & = & -2, \\ & & x_2 & + & x_3 & & & = & 3, \\ & & & & & & 0 & = & 0. \end{array} \tag{6}$$

We can rewrite (6) in the form

$$\begin{array}{rcl} x_1 & = & x_3 - 2x_4 - 2, \\ x_2 & = & -x_3 + 3, \\ x_3 & = & x_3, \\ x_4 & = & x_4. \end{array}$$

In this system, we can assign $x_3$ and $x_4$ at will, and then $x_1$ and $x_2$ are completely determined. Replacing $x_3$ by $s$ and $x_4$ by $t$, we can rewrite this system as

$$\begin{array}{rcl} x_1 & = & s - 2t - 2, \\ x_2 & = & -s + 3, \\ x_3 & = & s, \\ x_4 & = & t, \end{array}$$

where the parameters $s$ and $t$ range over the real numbers. We can write the solution in vector form as

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{7}$$

It is NOT the case that the solution set

$$S = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^n : \ (x_1, x_2, \ldots, x_n) \text{ satisfies (4) } \}$$

is a linear subspace of $\mathbb{R}^4$. However, we can write

$$S = \begin{pmatrix} -2 \\ 3 \\ 0 \\ 0 \end{pmatrix} + W,$$

where $W$ is the solution set to the "associated" homogeneous linear system considered in Example 1. Like all solution sets to homogeneous linear systems, $W$ is a linear subspace, and we can regard $S$ as an *affine subspace* of $\mathbb{R}^4$ which is parallel to the linear subspace $W$.