3. Examples of Complex Analytic Functions

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We now focus on various examples of complex analytic functions, starting with the rational functions, then continuing on to the exponential and logarithm functions, and finally the trigonometric functions and their inverses. Yet other examples of complex analytic functions come from the theory of ordinary differential equations.

The complex analytic functions we construct will give conformal maps from one region of the complex plane to another, thereby providing important cases in which we can solve for the steady-state distribution of temperature in a given region of the plane.

1 Rational functions

The simplest complex analytic functions are the rational functions. These are the functions

$$R(z) = \frac{P(z)}{Q(z)}, \qquad (1)$$

where

$$P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$$
$$\text{and} \quad Q(z) = b_n z^n + b_{n-1} z^{n-1} + \cdots + b_1 z + b_0$$

are polynomials with complex coefficients

$$a_m,\ a_{m-1},\ \ldots,\ a_1,\ a_0, \quad b_n,\ b_{n-1},\ \ldots,\ b_1,\ b_0,$$

and we make the assumptions that $a_m \neq 0$ and $b_n \neq 0$. We also assume that P and Q do not have any common factors. The function $R(z)$ is differentiable at every point of $\mathbb{C} - S$, where S is the finite set of points within $\mathbb{C}$ at which the denominator $Q(z)$ vanishes. The maximum of the two degrees, the degree of the numerator and the degree of the denominator, is called the *degree* of the rational function $R(z)$.

The sum of two rational functions is also rational function, as is the product. Indeed, one can check that with these operations of addition and multiplication, the space of rational functions satisfies the field axioms, and we denote this field by $\mathbb{C}(X)$. This is an important example of a field, which can be added to the earlier ones $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$.

The Fundamental Theorem of Algebra allows us to factor the polynomials $P(z)$ and $Q(z)$,

$$P(z) = a_m(z-z_1)^{p_1}(z-z_2)^{p_2}\cdots(z-z_r)^{p_r}$$
$$\text{and} \quad Q(z) = b_n(z-\zeta_1)^{q_1}(z-\zeta_2)^{q_2}\cdots(z-\zeta_s)^{q_s},$$

where the exponents denote the multiplicities of the roots, and this provides us with the first of the two important canonical forms for a rational function:

$$R(z) = \frac{a_m}{b_n}\frac{(z-z_1)^{p_1}(z-z_2)^{p_2}\cdots(z-z_r)^{p_r}}{(z-\zeta_1)^{q_1}(z-\zeta_2)^{q_2}\cdots(z-\zeta_s)^{q_s}}. \tag{2}$$

The zeros $z_1,\ldots,z_r$ of the numerator are called the *zeros* of the rational function, while the zeros $\zeta_1,\ldots,\zeta_s$ of the denominator are called its *poles*. The *order* of a zero or pole is its multiplicity as a root of either the numerator or the denominator. Poles of order one are said to be *simple*.

We say that a rational function $R(z)$ is *proper* if

$$m = \deg P(z) \le n = \deg Q(z),$$

and *strictly proper* if $m < n$. If $R(z)$ is not strictly proper, we can divide by the denominator to obtain

$$R(z) = P_1(z) + R_1(z),$$

where $P_1(z)$ is a polynomial and the remainder $R_1(z)$ is a strictly proper rational function.

Example 1. Suppose that

$$R(z) = \frac{z^3 - 2z^2 - 7z + 21}{z^2 - 9}.$$

This rational function is not strictly proper so we can divide numerator by denominator to obtain

$$R(z) = z - 2 + R_1(z), \quad \text{where the remainder} \quad R_1(z) = \frac{2z+3}{z^2-9}$$

is a strictly proper rational function.

Strictly proper rational functions have a second canonical form, called the *partial fraction decomposition*. This is extremely useful in calculating integrals of rational functions, as you may remember from calculus.

Theorem. *If the rational function $R(z) = P(z)/Q(z)$ is strictly proper, that is* *if* $\deg P < \deg Q$*, then* $R(z)$ *has a partial fraction decomposition*

$$\begin{aligned} R(z) = \frac{A_{1,0}}{(z-\zeta_1)^{q_1}} + \cdots + \frac{A_{1,q_1-1}}{(z-\zeta_1)} \\ + \frac{A_{2,0}}{(z-\zeta_2)^{q_2}} + \cdots + \frac{A_{2,q_2-1}}{(z-\zeta_2)} + \cdots \\ + \frac{A_{s,0}}{(z-\zeta_s)^{q_s}} + \cdots + \frac{A_{s,q_s-1}}{(z-\zeta_s)}. \end{aligned} \tag{3}$$

A proof of this theorem can be found on page 107 of Saff and Snider [4]. What you should focus on is how to calculate the partial fraction decomposition of a given rational function. Note that if all the poles of r are simple, that is all the roots of the denominator have multiplicity one, then (3) simplifies to

$$R(z) = \frac{A_1}{z-\zeta_1} + \frac{A_2}{z-\zeta_2} + \cdots + \frac{A_n}{z-\zeta_n}. \tag{4}$$

In this simpler case, one can find the coefficients in the partial fraction expansion by the formula

$$A_i = \lim_{z\to\zeta_i} (z-\zeta_i) R(z). \tag{5}$$

Example 2. We can write the rational function

$$R(z) = \frac{2z+3}{z^2-9} \quad \text{as} \quad R(z) = \frac{A_1}{z-3} + \frac{A_2}{z+3},$$

where use of (5) gives

$$\begin{aligned} A_1 = \lim_{z\to 3}(z-3)R(z) = \lim_{z\to 3}\frac{2z+3}{z+3} = \frac{9}{6} = \frac{3}{2}, \\ A_2 = \lim_{z\to -3}(z+3)R(z) = \lim_{z\to -3}\frac{2z+3}{z-3} = \frac{-3}{-6} = \frac{1}{2}. \end{aligned}$$

We conclude therefore that

$$R(z) = \frac{2z+3}{z^2-9} = \frac{3/2}{z-3} + \frac{1/2}{z+3}.$$

In the more general case, one needs to use the somewhat more complicated formula

$$A_{i,j} = \lim_{z\to\zeta_i}\left(\frac{1}{j!}\frac{d^j}{dz^j}\left[(z-\zeta_i)^{q_i}R(z)\right]\right). \tag{6}$$

Example 3. We can write the rational function

$$R(z) = \frac{z^2+2z+3}{(z-3)^3} \quad \text{as} \quad R(z) = \frac{A_{1,0}}{(z-3)^3} + \frac{A_{1,1}}{(z-3)^2} + \frac{A_{1,2}}{(z-3)},$$

where it follows from (6) that

$$\begin{aligned}
A_{1,0} &= \lim_{z\to 3}(z-3)^3 R(z) = \lim_{z\to 3}(z^2+2z+3) = 18,\\
A_{1,1} &= \lim_{z\to 3}\left(\frac{d}{dz}(z-3)^3 R(z)\right) = \lim_{z\to 3}\left(\frac{d}{dz}(z^2+2z+3)\right) = \lim_{z\to 3}(2z+2) = 8,\\
A_{1,2} &= \lim_{z\to 3}\left(\frac{1}{2}\frac{d^2}{dz^2}(z-3)^3 R(z)\right) = \lim_{z\to 3}\left(\frac{1}{2}\frac{d^2}{dz^2}(z^2+2z+3)\right) = \lim_{z\to 3}(1) = 1.
\end{aligned}$$

We conclude therefore that

$$R(z) = \frac{z^2+2z+3}{(z-3)^3} = \frac{18}{(z-3)^3} + \frac{8}{(z-3)^2} + \frac{1}{(z-3)}.$$

As pointed out in §1.4 of Chapter 2 in Ahlfors [1], it is quite convenient to regard the argument z of the rational functions $R(z)$, as well as the values of $R(z)$. as ranging over the extended complex plane $\mathbb{C}\cup\{\infty\}$. In considering the extension to $\mathbb{C}\cup\{\infty\}$, we make use of a new variable $w = 1/z$ which is well-behaved near ∞. Suppose that the numerator and denominator in the rational function have the same degree,

$$R(z) = \frac{a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0}{b_n z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0}.$$

Then replacing z by $1/w$ yields

$$\begin{aligned}
R(z) = \frac{a_n(1/w)^n + a_{n-1}(1/w)^{n-1} + \cdots + a_1(1/w) + a_0}{b_n(1/w)^n + b_{n-1}(1/w)^{n-1} + \cdots + b_1(1/w) + b_0}&\\
= \frac{a_n + a_{n-1}w + \cdots + a_1 w^{n-1} + a_0 w^n}{b_n + b_{n-1}w + \cdots + b_1 w^{n-1} + b_0 w^n}&.
\end{aligned}$$

Setting $z = \infty$ is the same as setting $w = 0$, so we let R denote also the extension of the rational function R to $\mathbb{C}\cup\{\infty\}$,

$$R(\infty) = \frac{a_n}{b_n}.$$

More generally, the reader can easily check that if

$$R(z) = \frac{P(z)}{Q(z)}, \quad \text{with} \quad \deg P = m \quad \text{and} \quad \deg Q = n,$$

then R has a zero of order $n-m$ at ∞ if $m < n$ and a pole of order $m-n$ at ∞ if $m > n$. In particular, strictly proper rational functions have zeros at ∞.

Similarly, we regard the value of $R(z)$ at a pole ζ_i as ∞. With this convention we can regard R as a map

$$R : \mathbb{C}\cup\{\infty\} \longrightarrow \mathbb{C}\cup\{\infty\}. \tag{7}$$

It is often helpful to use stereographic projection to regard this as defining a map from S^2 to itself. It turns out that if R has degree n there will be n solutions to the equation $R(z) = c$ for most choices of $c \in \mathbb{C} \cup \{\infty\}$.

Example 4. Let us reconsider the rational function of Example 1,

$$R(z) = \frac{z^3 - 2z^2 - 7z + 21}{z^2 - 9} = z - 2 + \frac{2z + 3}{z^2 - 9}.$$

As in Example 2, we can also write

$$R(z) = z - 2 + \frac{3/2}{z - 3} + \frac{1/2}{z + 3},$$

so R has simple poles at 3 and -3. Replacing z by $1/w$ yields,

$$R(z) = \frac{1}{w} - 2 + w\frac{2 + 3w}{1 - 9w^2},$$

so R also has a simple pole at ∞. Thus there are exactly three elements of $\mathbb{C} \cup \{\infty\}$ which solve the equation $R(z) = \infty$, namely 3, -3 and ∞.

2 Linear fractional transformations

Of central importance among the rational functions are the *linear fractional transformations* or *Möbius transformations*, the rational functions of degree one:

$$w = T(z) = \frac{az + b}{cz + d}, \tag{8}$$

where a, b, c and d are complex numbers such that $ad - bc \neq 0$. By solving for z as a function of w, we can show that any linear fractional transformation T has an inverse,

$$z = T^{-1}(w) = \frac{dw - b}{-cw + a},$$

as one verifies by a calculation. Thus linear fractional transformations are always one-to-one and onto as maps from $\mathbb{C} \cup \{\infty\}$ to itself. Indeed they are the only complex analytic functions with complex analytic inverses from $\mathbb{C} \cup \{\infty\}$ to itself.

Moreover, one easily checks that the composition of two linear fractional transformations is another linear fractional transformation. Indeed, composition makes the space of linear fractional transformations into what algebraists call a group.

When $c = 0$ and $d = 1$, the linear fractional transformation reduces to a *complex linear transformation*

$$z \mapsto w = T(z) = az + b.$$

The complex linear transformations include the

$$\begin{cases} \text{translations} & z \mapsto w = T(z) = z + b, \\ \text{rotations} & z \mapsto w = T(z) = e^{i\theta} z, \\ \text{and expansions and contractions} & z \mapsto w = T(z) = \rho z, \end{cases}$$

where θ and ρ are real numbers, ρ being nonzero. When $a = d = 0$ and $b = c = 1$, the linear fractional transformation reduces to an *inversion*

$$w = T(z) = \frac{1}{z}.$$

Here T is a reflection in the circle

$$z \mapsto \frac{1}{\bar{z}}, \quad re^{i\theta} \mapsto \frac{1}{r} e^{i\theta},$$

followed by conjugation.

The following Proposition is quite useful in determining the properties of a given linear fractional transformation. Linear fractional transformations will become increasingly important as the course progresses, and will be studied in more detail in 122B; see §7.3 of [4] or Chapter 7, § 5 of [3].

Proposition. *a. Any linear fractional transformation is the composition of complex linear maps and inversions. b. Any linear fractional transformation takes circles and lines to circles and lines. c. Given any three distinct points z_1, z_2 and z_3 of $\mathbb{C} \cup \{\infty\}$, there is a unique linear fractional transformation T such that*

$$T(z_1) = 0, \quad T(z_2) = 1 \quad \text{and} \quad T(z_3) = \infty. \tag{9}$$

Sketch of proof of a: Note that the transformation (8) is unchanged if we make the replacements

$$a \mapsto \lambda a, \quad b \mapsto \lambda b, \quad c \mapsto \lambda c, \quad d \mapsto \lambda d,$$

where $\lambda \neq 0$. So we can assume without loss of generality that $ad - bc = 1$. Moreover, we can assume without generality that $c \neq 0$, because when $c = 0$, the transformation is clearly linear. We can then factor the map

$$z' = T(z) = \frac{az + b}{cz + d}$$

into a composition of four transformations

$$z_1 = z + \frac{d}{c}, \quad z_2 = c^2 z_1, \quad z_3 = -\frac{1}{z_2}, \quad z' = z_3 + \frac{a}{c}. \tag{10}$$

To see this, note that

$$z_2 = c(cz + d), \quad z_3 = -\frac{1}{c(cz + d)},$$
$$z' = \frac{a(cz + d)}{c(cz + d)} - \frac{1}{c(cz + d)} = \frac{acz + ad - 1}{c(cz + d)} = \frac{az + b}{cz + d}.$$

Each of the transformations in (10) is either complex linear or an inversion, so the Proposition is proven.

Sketch of proof of b: Complex linear transformations take circles and lines to circles and lines, so we need only check that inversions take circles and lines to circles and lines. But the equation of a circle can be written as

$$A(x^2 + y^2) + \mathrm{Re}[(C - iD)(x + iy)] + B = 0,$$

for some choice of real constants A, B, C and D. We can rewrite this as

$$Az\bar{z} + \mathrm{Re}[(C - iD)z] + B = 0.$$

If we replace z by $1/w$, this becomes

$$\frac{A}{w\bar{w}} + \mathrm{Re}\left[\frac{(C - iD)}{w}\right] + B = 0$$

or

$$A + \mathrm{Re}\left[(C - iD)\bar{w}\right] + Bw\bar{w} = 0,$$

which is once again the equation for a circle.

Sketch of proof of c: If none of the three points is infinity, the linear fractional transformation satisfying (9) is

$$T(z) = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}. \tag{11}$$

One needs to adjust this formula appropriately when one of the three points is ∞.

Example. Suppose that we want a linear fractional transformation T such that

$$T(-i) = 0, \quad T(1) = 1, \quad T(i) = \infty.$$

Use of the formula (11) yields

$$T(z) = \frac{(z + i)(1 - i)}{(z - i)(1 + i)} = \frac{z + i}{iz + 1}.$$

This transformation takes the unique circle through $-i$, 1 and i, which turns out to be the unit circle, to the straight line through 0, 1 and ∞, which is the x-axis. Moreover, $T(0) = i$. From this one can conclude that T takes the unit disk

$$D = \{z \in \mathbb{C} : |z| < 1|]$$

to the upper half-plane

$$H = \{x + iy \in \mathbb{C} : y > 0\}.$$

3 Exponential and trigonometric functions

Earlier, we defined the complex exponential function in terms of real-valued functions as

$$w = \exp(z) = e^z = e^x(\cos y + i \sin y), \tag{12}$$

and used identities from trigonometry to show that

$$e^{z_1+z_2} = e^{z_1}e^{z_2},$$

as well as to show that the exponential function is periodic of period $2\pi i$:

$$\exp(z + 2\pi k i) = \exp(z), \quad \text{for } k \in \mathbb{Z}. \tag{13}$$

Moreover, writing

$$e^z = u(x, y) + iv(x, y), \quad \text{where} \quad u(x, y) = e^x \cos y \quad \text{and} \quad v(x, y) = e^x \sin y$$

and using the Cauchy-Riemann equations, we find that

$$\frac{d}{dz}(e^z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = e^x \cos y + ie^x \sin y = e^z. \tag{14}$$

Although the exponential function exp is certainly quite well-behaved on $\mathbb{C}$, it does not extend to a well-behaved function on the extended complex plane $\mathbb{C} \cup \{\infty\}$. Neither

$$\lim_{w\to 0} e^{1/w} \quad \text{nor} \quad \lim_{w\to 0} \frac{1}{e^{1/w}}$$

exists, and hence one says that $\exp(z)$ has an "essential singularity" at ∞. Indeed, one can show that $e^{1/w}$ takes on every nonzero complex number infinitely many times when w lies in any neighborhood $N(0; \varepsilon)$ of 0.

We would like to define a function which is inverse to exp on $\mathbb{C} - \{0\}$, but the periodicity makes it impossible to define an inverse which is a genuine function. Indeed, if $w = e^z$, it only follows from (12) that

$$w = re^{i\theta} = r(\cos\theta + i \sin\theta) = e^x(\cos y + i \sin y),$$

so y is one of the many possible values of the angular coordinate θ, all of these values differing by integer multiples of 2π. Thus we write

$$x = \text{Log}(r) = \text{Log}(|w|), \quad y = \theta = \arg(w),$$

where arg is the multiple-valued function which gives the various possible angular coordinates θ of w, and

$$z = x + iy = \text{Log}(|w|) + i\arg(w).$$

In this way, we are led to define the multiple-valued function log by

$$\log(w) = \text{Log}(|w|) + i\arg(w),$$

where $\text{Log}(|w|)$ is the usual natural logarithm of the nonzero real number $|w|$. It follows from (13) that the various values of the multiple-valued function log differ by integer multiples of 2π.

Dangerous curve: It is important to note that a multiple-valued function is NOT a genuine function as studied in the rest of mathematics. A genuine function can only assume one value for a given choice of argument. Nevertheless, the notion of multiple-valued function is commonly used within the theory of complex variables, because it is quite useful. In more advanced treatments of complex variables, one tries to eliminate multiple-valued functions on open subsets of the complex plane by passing instead to genuine single-valued functions defined on "Riemann surfaces" over the open subsets (see [2]).

As we saw earlier, we can eliminate the ambiguity in the logarithm by replacing $\mathbb{C} - \{0\}$ by the smaller domain

$$D = \mathbb{C} - \{x + iy \in \mathbb{C} : x \leq 0\}, \tag{15}$$

and choosing a *branch* of the muliple-valued function log that is single-valued in D. We do this by first letting $\text{Arg}(w)$ be the value of the angular coordinate θ which lies in the interval $(-\pi, \pi]$ and then define

$$\text{Log} : D \longrightarrow \mathbb{C} \quad \text{by} \quad \text{Log}(w) = \text{Log}(|w|) = i\text{Arg}(w).$$

Choosing a branch does give us a genuine single-valued complex analytic logarithm function, and as we saw when discussing the Cauchy-Riemann equations,

$$\frac{d}{dw}\text{Log}(w) = \frac{1}{w}.$$

However, this destroys some of the nice properties one might hope for the logarithm; for example, although

$$\log(z_1 z_2) = \log(z_1) + \log(z_2),$$

where equality means that the values taken by the multiple-valued functions on the two sides are the same, it is not true that

$$\text{Log}(z_1 z_2) = \text{Log}(z_1) + \text{Log}(z_2)$$

in general.

One of the most useful applications of the multiple-valued logarithm function is in defining arbitrary powers of a complex number. The definition is motivated by the fact that

$$z^n = \exp\left(n\log(z)\right), \quad \text{when} \quad n \in \mathbb{N}.$$

Definition. If $z \in \mathbb{C} - \{0\}$ and $\alpha \in \mathbb{C}$, we define z^α by

$$z^\alpha = \exp\left(\alpha\log(z)\right).$$

Example 1. Let us consider the possible values of

$$i^{-2i} = \exp(-2i\log(i)).$$

To do this, we first note that

$$\log(i) = \log 1 + i\arg(i) = 0 + i\frac{\pi}{2} + 2\pi k i, \quad \text{for } k \in \mathbb{Z},$$

so

$$i^{-2i} = \exp(\pi + 4\pi k), \quad \text{for } k \in \mathbb{Z}.$$

This example illustrates that just like logarithms, the complex power z^α can have infinitely many different values. Although square roots can have only two different values, they provide important examples of multiple-valued functions. Later, when we want to integrate square roots along contours, we will need to choose a particular branch for the integrals to be well-defined. We provide a few examples to illustrate how this is done:

Example 2. We construct two branches f_1 and f_2 of the square root function

$$w = \pm\sqrt{z},$$

by setting

$$f_1(z) = \exp\left(\frac{1}{2}\mathrm{Log}(z)\right), \qquad f_2(z) = -\exp\left(\frac{1}{2}\mathrm{Log}(z)\right),$$

for $z \in D$, where D is the domain defined by (15), the complex plane minus the negative x-axis. One can imagine two copies D_1 and D_2 of D (called *sheets*) which are glued together along the negative x-axis. As one crosses the negative x-axis one passes from sheet to the other. The union of the two sheets can be regarded as a "Riemann surface" on which the square root function is a genuine single-valued function.

Example 3. Suppose that we want to construct a branch of the multiple-valued function

$$w = \pm\sqrt{1 - z^2}. \tag{16}$$

To do this, we note that the multiple-valued function can be defined by

$$w = \exp\left(\frac{1}{2}\log(1 - z^2)\right),$$

and that $\log(1 - z^2)$ fails to be defined when $z = \pm 1$. We can then construct two branches

$$w = f_1(z) = \exp\left(\frac{1}{2}\mathrm{Log}\left(1 - z^2\right)\right)$$

and

$$w = f_2(z) = -\exp\left(\frac{1}{2}\mathrm{Log}\left(1 - z^2\right)\right),$$

which will be a single-valued functions defined so long as $1 - z^2$ does not lie on the negative part of the x-axis, that is, so long as z is not real with $|z| > 1$. One can imagine that the two branches of the multiple-valued function $\pm\sqrt{1 - z^2}$ are interchanged when one crosses $\mathbb{R} - [-1, 1]$.

Example 4. Similarly, we can construct branches of the multiple-valued function

$$w = \pm\sqrt{z^2 - 1} = \pm i\sqrt{1 - z^2},$$

which is not much different from the function considered in the previous example, but this time we seek branches are defined and single-valued outside the unit disk. For this, we note first that the multiple-valued function can also be written as

$$w = z \exp\left(\frac{1}{2}\log\left(1 - \frac{1}{z^2}\right)\right),$$

and in this representation $1 - (1/z)^2$ lies on the negative x-axis exactly when z is real and $|z| < 1$. Thus the functions

$$w = f_1(z) = z \exp\left(\frac{1}{2}\mathrm{Log}\left(1 - \frac{1}{z^2}\right)\right)$$

and

$$w = f_2(z) = -z \exp\left(\frac{1}{2}\mathrm{Log}\left(1 - \frac{1}{z^2}\right)\right)$$

will be a single-valued functions defined on the set

$$D = \mathbb{C} - \{z = x + iy \in \mathbb{C} : y = 0 \ \& \ -1 < x < 1\}.$$

In this case, one can imagine that the two branches of the multiple-valued function $\pm\sqrt{z^2 - 1}$ are interchanged when one crosses the interval $(-1, 1) \subseteq \mathbb{R}$.

Trigonometric functions. As mentioned before, we can use Euler's identity to express the trigonometric functions cosine and sine in terms of exponentials,

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i},$$

thereby achieving a simplifying unity between exponential and trigonometric functions. We can use these expressions to determine all of the usual properties of the trigonometric functions. For example, it follows from (14) that

$$\frac{d}{dz}(\sin z) = \cos z, \qquad \frac{d}{dz}(\cos z) = -\sin z.$$

Thus, it follows from the definition of derivative that if $f(z) = \sin z$,

$$1 = \cos(0) = f'(0) = \lim_{\Delta z \to 0} \frac{f(\Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\sin(\Delta z)}{\Delta z} = \lim_{h \to 0} \frac{\sin h}{h},$$

a limit familiar in first year calculus when h is assumed to be real.

Using the logarithm, we can also solve to find the inverses of the trigonometric functions. Thus if

$$z = \sin w = \frac{e^{iw} - e^{-iw}}{2i},$$

we can multiply through by e^{iw}, obtaining

$$2ie^{iw}z = \left(e^{iw}\right)^2 - 1 \quad \text{or} \quad \left(e^{iw}\right)^2 - 2ize^{iw} - 1 = 0.$$

Thus it follows from the quadratic formula that

$$e^{iw} = iz \pm \sqrt{1-z^2},$$

and we find that

$$w = \arcsin(z) = -i \log\left(iz \pm \sqrt{1-z^2}\right), \tag{17}$$

which is, of course, a multiple-valued function. Similarly, the inverse of the cosine is a mutiple-valued function

$$w = \arccos(z) = \frac{\pi}{2} - \arcsin(z) = \frac{\pi}{2} - i \log\left(iz \pm \sqrt{1-z^2}\right).$$

To choose a specific branch of the inverse trigonometric functions, we must choose a branch of the square root as well as of the logarithm. For any such branch, it follows from (17) that

$$\frac{d}{dz}(\arcsin(z)) = \frac{1}{\pm\sqrt{1-z^2}}.$$

Note that by Example 3, a single-valued branch of the derivative of the inverse of the sine can be defined over

$$D = \mathbb{C} - (\mathbb{R} - [-1,1]),$$

and we will later see that this enables us to construct a single-valued inverse of the sine itself over the same region D.

Similarly, we can find the inverses of the hyperbolic cosine; thus if

$$z = \cosh w = \frac{e^w + e^{-w}}{2},$$

we can multiply through by e^w, obtaining

$$2e^w z = (e^w)^2 + 1 \quad \text{or} \quad (e^w)^2 - 2ze^w + 1 = 0.$$

This time the quadratic formula, yields

$$e^w = \frac{2z \pm \sqrt{4z^2-4}}{2} = z \pm \sqrt{z^2-1}.$$

Thus we find that

$$w = \operatorname{arccosh}(z) = \log\left(z \pm \sqrt{z^2 - 1}\right)$$

Differentiation this time yields

$$\frac{d}{dz}(\operatorname{arccosh}(z)) = \frac{1}{\pm\sqrt{z^2 - 1}},$$

the derivative possessing a single-valued branch over the domain $\mathbb{C} - [-1, 1]$ by Example 4.

4 Steady-state temperature distributions

We now illustrate the use of the real and imaginary parts of the complex logarithm in determining the steady-state temperature distribution in several regions of the plane.

But first we must show that steady-state temperature is represented by a solution to the Laplace equation. We suppose that a region U of the (x, y)-plane contains a material that has homogeneous properties. We would like to determine the steady-state temperature within the region when the temperature is given on the boundary.

Let $u(x, y)$ be the temperature at the point (x, y). Then the flow of heat at (x, y) should be $\mathbf{V}(x, y)$, where

$$\mathbf{V}(x, y) = -(\kappa \nabla u)(x, y) = -\kappa \frac{\partial u}{\partial x}(x, y)\mathbf{i} - \kappa \frac{\partial u}{\partial x}(x, y)\mathbf{j}, \tag{18}$$

where κ is a positive constant, called the *thermal conductivity* of the material. We agree to set

$$\mathbf{V}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}.$$

If D is a region within U with smooth boundary ∂D, then it follows from the divergence theorem that

$$\int_{\partial D} (M\mathbf{i} + N\mathbf{j}) \cdot \mathbf{N} ds = \int\int_D \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy,$$

where $\mathbf{N}$ is the outward-pointing unit normal to ∂D, and s is the arc-length parameter along ∂D. Recall that the line integral on the left is calculated by means of a unit-speed parametrization $\gamma : [a, b] \to \partial D$ with $\gamma(a) = \gamma(b)$.

This line integral can be interpreted as the rate at which heat is flowing outward across ∂D. Thus if no heat is being created or destroyed within D, the line integral must vanish, and

$$\int\int_D \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy = 0.$$

If no heat is being created or destroyed anywhere within U, this double integral must vanish for every choice of $D \subseteq U$, and this can only happen if

$$\nabla \cdot \mathbf{V} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \equiv 0 \quad \text{on } U.$$

In the special case where $\mathbf{V}$ is given by (18), we conclude that the temperature u must satisfy Laplace's equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

or equivalently, u must be a harmonic function. (See §2.6 of [4] for a more leisurely discussion.)

Since harmonic functions are the real parts of complex analytic functions, we have a nice application of complex analytic functions: Their real parts can represent the steady-state temperature distribution in a homogeneous medium.

Example 1. Suppose we want to determine the steady state temperature distribution in the annular region

$$D = \{x + iy \in \mathbb{C} : 1 \leq x^2 + y^2 \leq e^2\},$$

given that the temperature on the circle $x^2 + y^2 = 1$ is 40 degrees and the temperature on the circle $x^2 + y^2 = e^2$ is 70 degrees.

To solve this problem, it is quite natural to use polar coordinates (r, θ). We need to find those harmonic functions which depend only on the radial coordinate r. This problem was solved in the discussion following Problem C in the Notes on Complex Analytic Functions, and we found that the only such harmonic functions are

$$u = a \text{ Log } r + b = a \text{ Log}|z| + b,$$

where a and b are real constants. The boundary conditions are that $u = 40$ when $r = 1$ and $u = 70$ when $r = e$. Thus

$$\begin{cases} a \text{ Log}(1) + b = b = 40, \\ a \text{ Log}(e) + b = a + b = 70, \end{cases}$$

and we find that $b = 40$ and $a = 30$, so

$$u = 30 \text{ Log } r + 40.$$

Example 2. Suppose we want to determine the steady state temperature distribution $u(z) = u(x, y)$ in the wedge

$$D = \{x + iy \in \mathbb{C} : x > 0, \ y < x, \ y > -x\},$$

given that the temperature on the ray $y = x, \ x > 0$ is 90 degrees and the temperature on the ray $y = -x, \ x > 0$ is 50 degrees. Once again it is natural to use polar coordinates (r, θ), and we seek those harmonic functions which depend only on the angular coordinate θ. This time the only such harmonic functions are

$$u = a\,\theta + b,$$

where a and b are real constants. The boundary conditions are that $u = 90$ when $\theta = \pi/4$ and $u = 50$ when $\theta = -\pi/4$. Thus

$$\begin{cases} a(\pi/4) + b = 90, \\ -a(\pi/4) + b = 50, \end{cases}$$

and we find that the solution to the linear equations is $a = 80/\pi$ and $b = 70$, so

$$u = \frac{80\,\theta}{\pi} + 70 \quad \text{or} \quad u = \frac{80}{\pi}\mathrm{Arg}(x+iy) + 70.$$

Example 3. Suppose we want to determine the steady state temperature distribution $u(z) = u(x,y)$ in the upper half-plane, given that the temperature along the real axis is given by

$$u(x,0) = \begin{cases} 90, & \text{for } x < 1, \\ 70, & \text{for } -1 < x < 1, \\ 50, & \text{for } x > 1. \end{cases}$$

In this case, it is natural to consider the temperature as given by a superposition of argument functions

$$u(z) = a + b_1 \mathrm{Arg}(z-1) + b_2 \mathrm{Arg}(z+1).$$

The boundary conditions in this case give

$$a + \pi b_1 + \pi b_2 = 90, \quad a + \pi b_1 = 70, \quad a = 50,$$

and we quickly find that the solution is

$$u(z) = 50 + \frac{20}{\pi}\mathrm{Arg}(z-1) + \frac{20}{\pi}\mathrm{Arg}(z+1).$$

Proposition. *Suppose that U and V are open subsets of $\mathbb{C}$. If $f : U \to V$ is complex analytic, and $u : V \to \mathbb{R}$ is harmonic, then $u \circ f : U \to \mathbb{R}$ is harmonic.*

Sketch of proof: First reduce to the case where U and V are open balls. Then one can use the Poincaré Lemma to construct a harmonic conjugate v to u and

$$g = u + iv : V \to \mathbb{C}$$

is complex analytic. Then it follows from the chain rule that $g \circ f : U \to \mathbb{C}$ is also complex analytic, and hence

$$u \circ f = \mathrm{Re}(g \circ f) : U \to \mathbb{R}$$

is harmonic.

Example 4. We can now use the preceding example together with the linear fractional transformation

$$T(z) = \frac{z+i}{iz+1},$$

described at the end of §2 to determine the steady-state temperature in the unit disk,

$$D = \{x + iy \in \mathbb{C} : x^2 + y^2 \leq 1\}$$

which satisfies the boundary conditions that the temperature be 50 degrees on the part C_1 of the circle $x^2 + y^2 = 1$ lying in the first quadrant, 70 degrees on the part C_2 of the circle $x^2 + y^2$ lying below the x-axis, and 90 degrees on the remaining quarter circle C_3.

It follows from the Proposition that if $u(z)$ is the harmonic function of Example 3, then $u \circ T$ is also harmonic. Moreover, T takes D to the upper half-plane, takes the counterclockwise unit circle to the x-axis traversed in the positive direction, and satisfies

$$T(-1) = -1, \quad T(1) = 1, \quad T(i) = \infty.$$

Hence it takes C_1, C_2 and C_3 to the segments $x > 1$, $-1 < x < 1$ and $x < -1$ along the x-axis, and the desired function is

$$(u \circ T)(z) = 50 + \frac{20}{\pi}\mathrm{Arg}(T(z) - 1) + \frac{20}{\pi}\mathrm{Arg}(T(z) + 1).$$

References

[1] Lars V. Ahlfors, *Complex analysis*, 3rd ed., McGraw-Hill, New York, 1979.

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[3] S. Lang, *Complex analysis*, 3rd ed., Springer, New York, 1993.

[4] E. B. Saff and A. D. Snider, *Fundamentals of complex analysis*, 3rd ed., Pearson, Upper Saddle River, NJ, 2003.