5.8 Let $S = \{\varnothing, \{\varnothing\}\}$. Determine whether each of the following is True or False. Explain your answers.

(a) $\varnothing \subseteq S$ (b) $\varnothing \in S$ (c) $\{\varnothing\} \subseteq S$ (d) $\{\varnothing\} \in S$

5.9 Fill in the blanks in the following proof of Theorem 5.13(a).

THEOREM: Let A be a subset of U. Then $A \cup (U \backslash A) = U$.

Proof: If $x \in A \cup (U \backslash A)$, then $x \in$ ________ or $x \in$ ________. Since both A and $U \backslash A$ are subsets of U, in either case we have ________. Thus ________ $\subseteq$ ________.

On the other hand, suppose that $x \in$ ________. Now either $x \in A$ or $x \notin A$. If $x \notin A$, then $x \in$ ________. In either case $x \in$ ________. Hence ________ $\subseteq$ ________. ♦

5.10 Fill in the blanks in the proof of the following theorem.

THEOREM: $A \subseteq B$ iff $A \cup B = B$.

Proof: Suppose that $A \subseteq B$. If $x \in A \cup B$, then $x \in A$ or $x \in$ ________. Since $A \subseteq B$, in either case we have $x \in B$. Thus ________ $\subseteq$ ________. On the other hand, if $x \in$ ________, then $x \in A \cup B$, so ________ $\subseteq$ ________. Hence $A \cup B = B$.

Conversely, suppose that $A \cup B = B$. If $x \in A$, then $x \in$ ________. But $A \cup B = B$, so $x \in$ ________. Thus ________ $\subseteq$ ________. ♦

5.11 Fill in the blanks in the proof of the following theorem.

THEOREM: $A \subseteq B$ iff $A \cap B = A$.

Proof: Suppose that $A \subseteq B$. If $x \in A \cap B$, then clearly $x \in A$. Thus $A \cap B \subseteq A$. On the other hand, ____.
Thus $A \subseteq A \cap B$, and we conclude that $A \cap B = A$.

Conversely, suppose that $A \cap B = A$. If $x \in A$, then ____.
Thus $A \subseteq B$. ♦

5.12 Suppose you are to prove that set A is a subset of set B. Write a reasonable beginning sentence for the proof, and indicate what you would have to show in order to finish the proof.

5.13 Suppose you are to prove that sets A and B are disjoint. Write a reasonable beginning sentence for the proof, and indicate what you would have to show in order to finish the proof.

5.14 Which statement(s) below would enable one to conclude that $x \in A \cup B$?

(a) $x \in A$ and $x \in B$. (b) $x \in A$ or $x \in B$.
(c) If $x \in A$, then $x \in B$. (d) If $x \notin A$, then $x \in B$.

6.2 Mark each statement True or False. Justify each answer.

(a) If $\mathcal{P}$ is a partition of set S, then we can obtain a relation R on S by defining $x \mathrm{R} y$ iff x and y are in the same piece of the partition.
(b) In any relation R on a set S, we always have $x\mathrm{R}x$ for all $x \in S$.
(c) If R is a relation on S, then $\{y \in S\colon y\mathrm{R}x\}$ determines a partition of S.
(d) If $\mathcal{P}$ is a partition of S and $x \in S$, then $x \in A$ for some $A \in \mathcal{P}$.

6.3 Using Definition 6.1, show that $(a, a) = \{\{a\}\}$.

6.4 Using Definition 6.1, show that $\{a\} \times \{a\} = \{\{\{a\}\}\}$.

6.5 Using Definition 6.1, find $(2,3) \cap (3,2)$.

6.6 Let A be any set and let $B = \varnothing$. What can you conclude about $A \times B$?

6.7 Let $A = \{a\}$ and $B = \{1,2,3\}$. List all possible relations between A and B.

6.8 Let $A = \{a, b\}$.

(a) How many elements are in the set $A \times A$?
(b) How many possible relations are there on set A?
(c) How many possible relations are there on the set $\{a, b, c\}$?

6.9 Fill in the blanks in the proof of the following theorem. ☆

THEOREM: $(A \cap B) \times C = (A \times C) \cap (B \times C)$

Proof: Let $(x, y) \in (A \cap B) \times C$. Then $x \in$ ______ and $y \in$ ______. Since $x \in A \cap B$, $x \in$ ______ and $x \in$ ______. Thus $(x, y) \in$ ______ and $(x, y) \in$ ______. Hence $(x, y) \in (A \times C) \cap (B \times C)$, so ______ $\subseteq$ ______.

On the other hand, suppose that $(x, y) \in$ ______. Then $(x, y) \in$ ______ and $(x, y) \in$ ______. Since $(x, y) \in A \times C$, $x \in$ ______ and $y \in$ ______. Since $(x, y) \in B \times C$, ______ and ______. Thus $x \in A \cap B$, so ______ $\in$ ______ and ______ $\subseteq$ ______. ♦

6.10 Prove or give a counterexample.

(a) $A \times B = B \times A$
(b) $(A \cup B) \times C = (A \times C) \cup (B \times C)$
(c) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$
(d) $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$

6.11 Determine which of the three properties (reflexive, symmetric, and transitive) apply to each relation.

(a) Let R be the relation on $\mathbb{N}$ given by $x \mathrm{R} y$ iff x divides y. ☆
(b) Let X be a set and let R be the relation "$\subseteq$" defined on subsets of X.
(c) Let S be the set of people in the school. Define R on S by $x\mathrm{R}y$ iff "x likes y."

7.6 Let $A \subseteq \mathbb{R}$ and define $f: A \to \mathbb{R}$ as given below. In each case describe a set A so that f is injective on A. Make A as large as possible.

(a) $f(x) = (x+3)^2 - 5$
(b) $f(x) = |2x-1|$
(c) $f(x) = \sin x$

7.7 Classify each function as injective, surjective, bijective, or none of these.

(a) $f: \mathbb{N} \to \mathbb{N}$ defined by $f(n) = n + 3$ ☆
(b) $f: \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = n - 5$
(c) $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 - x$ ☆
(d) $f: [1, \infty) \to [0, \infty)$ defined by $f(x) = x^3 - x$
(e) $f: \mathbb{N} \to \mathbb{Z}$ defined by $f(n) = n^2 - n$
(f) $f: [3, \infty) \to [5, \infty)$ defined by $f(x) = (x-3)^2 + 5$
(g) $f: \mathbb{N} \to \mathbb{Q}$ defined by $f(n) = 1/n$

7.8 (a) Let S be the set of all circles in the plane. Define $f: S \to [0, \infty)$ by $f(C)$ = the area of C, for all $C \in S$. Is f injective? Is f surjective?
(b) Let T be the set of all circles in the plane that are centered at the origin. Define $g: T \to [0, \infty)$ by $g(C)$ = the area of C, for all $C \in T$. Is g injective? Is g surjective?

7.9 Consider the following theorem:

The function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 5x + 3$ is injective.

Indicate what, if anything, is wrong with each of the following "proofs."

(a) Let $x_1, x_2 \in \mathbb{R}$ and suppose $x_1 = x_2$. Then $5x_1 = 5x_2$ and $5x_1 + 3 = 5x_2 + 3$, so $f(x_1) = f(x_2)$. Thus f is injective. ☆
(b) Let $x_1, x_2 \in \mathbb{R}$ and suppose $f(x_1) = f(x_2)$. We must prove that $x_1 = x_2$. Now $f(x_1) = 5x_1 + 3$ and $f(x_2) = 5x_2 + 3$. Since $x_1 = x_2$, we have $5x_1 + 3 = 5x_2 + 3$. It follows that $5x_1 = 5x_2$ and $x_1 = x_2$. Thus f is injective.
(c) Let $x_1, x_2 \in \mathbb{R}$ and suppose $x_1 \neq x_2$. Then $5x_1 \neq 5x_2$ and $5x_1 + 3 \neq 5x_2 + 3$, so $f(x_1) \neq f(x_2)$. It follows that $x_1 = x_2$ whenever $f(x_1) = f(x_2)$, and f is injective.
(d) Let $x_1, x_2 \in \mathbb{R}$ and suppose $f(x_1) \neq f(x_2)$. Thus $5x_1 + 3 \neq 5x_2 + 3$ and $5x_1 \neq 5x_2$, so $x_1 \neq x_2$. It follows that $f(x_1) = f(x_2)$ only if $x_1 = x_2$, and f is injective.
(e) We have $f(1) = 8$ and $f(2) = 13$, so if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$. It follows that f is injective.
(f) Let $x_1, x_2 \in \mathbb{R}$ and suppose $f(x_1) = f(x_2)$. Then $5x_1 + 3 = 5x_2 + 3$ and $5x_1 = 5x_2$, so $x_1 = x_2$. Thus f is injective.

7.10 In each part, find a function $f: \mathbb{N} \to \mathbb{N}$ that has the desired properties.

(a) surjective, but not injective
(b) injective, but not surjective
(c) neither surjective nor injective
(d) bijective