Key Linear Independence Theorems

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John Douglas Moore

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Linear Dependence Lemma. *Suppose that* $(\mathbf{v}_1, \ldots, \mathbf{v}_m)$ *is a linearly dependent list of vectors in a vector space* V *over a field* F*, and that* $\mathbf{v}_1 \neq \mathbf{0}$*. Then there exists* $j \in \{2, \ldots, m\}$ *such that*

$$\mathbf{v}_j \in Span(\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}).$$

Moreover,

$$Span(\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots \mathbf{v}_m) = Span(\mathbf{v}_1, \ldots, \mathbf{v}_m).$$

Assuming this lemma, we prove the MAIN RESULT of Chapter 1 in the text:

Replacement Theorem. *If* V *is a vector space over a field* F*,* $(\mathbf{u}_1, \ldots, \mathbf{u}_m)$ *is a linearly independent list of elements from* V*, and* V *is the span of a list* $(\mathbf{w}_1, \ldots, \mathbf{w}_n)$*, then* $m \leq n$*.*

Idea of proof: One by one replace elements of the spanning list by elements of the linear independent list, renormalizing to the same size by means of the Linear Dependence Lemma.

Definition. A *basis* for a vector space V is a list $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ which is linearly independent and spans V.

Corollary. *If* V *is a vector space over a field* F*, Any two finite bases for* V *have the same number of elements.*

Definition. A vector space V over a field F is finite-dimensional if it has a basis which has finitely many elements. The *dimension* of a finite-dimensional vector space V is the number of elements in any of its bases. We let $\dim(V)$ denote the dimension of V.

Theorem. *Every spanning list in a finite-dimensional vector space* V *can be reduced to a basis.*

Idea of proof: Start with a spanning set and throw away elements until you have a basis. As long as you don't have a basis, the Linear Dependence Lemma says that you can throw something away.

Theorem. *Every linearly independent list in a finite-dimensional vector space V can be extended to a basis.*

Idea of proof: Suppose that $B = (\mathbf{u}_1, \dots, \mathbf{u}_m)$ is a linearly independent list. Since V is finite-dimensional, we can write $V = \text{span}(\mathbf{w}_1, \dots, \mathbf{w}_n)$. One by one, add the $\mathbf{w}_i$'s to the list B, throwing away any additions that make the list linearly dependent (by means of the Linear Dependence Lemma).

Definition. Suppose that U and W are subspaces of a vector space V. We say that V is the *direct sum* of U and W, and we write $V = U \oplus W$, if

1. $U \cap W = \{\mathbf{0}\}$, and
2. every element $\mathbf{v} \in V$ is of the form $\mathbf{u} + \mathbf{w}$ where $\mathbf{u} \in U$ and $\mathbf{w} \in W$.

Theorem. *Let U be a subspace of a finite-dimensional vector space V. Then there is a subspace W of V such that $V = U \oplus W$.*

Idea of proof: Choose a basis $(\mathbf{u}_1, \dots, \mathbf{u}_m)$ for U and extend it to a basis $(\mathbf{u}_1, \dots, \mathbf{u}_m, \mathbf{w}_{m+1}, \dots, \mathbf{w}_n)$ of V. Then $(\mathbf{w}_{m+1}, \dots, \mathbf{w}_n)$ is a basis for $W = \text{span}(\mathbf{w}_{m+1}, \dots, \mathbf{w}_n)$ and $V = U \oplus W$.