The Matrix of a Linear Map

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Suppose that $\beta = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ is a basis for a vector space V. By Proposition 2.8 in the text, if $\mathbf{v} \in V$, we can write

$$\mathbf{v} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \dots & \mathbf{v}_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ x_n \end{pmatrix}, \tag{1}$$

where we are using the usual matrix multipliation on the right. We say that $(x_1, \dots, x_n)$ are the *coordinates* of $\mathbf{v}$ with respect to the basis $(\mathbf{v}_1, \dots, \mathbf{v}_n)$.

For example if $V = \mathbb{R}^n$, we could take the standard basis $(\mathbf{e}_1, \dots, \mathbf{e}_n)$, where

$$\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \\ \cdot \\ 0 \end{pmatrix}, \quad \mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \\ \cdot \\ 0 \end{pmatrix}, \quad \cdots, \quad \mathbf{e_n} = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ 1 \end{pmatrix}.$$

Then if $\mathbf{x} \in \mathbb{R}^n$, we can write

$$\mathbf{x} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 \dots & \mathbf{e}_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ x_n \end{pmatrix}.$$

In this case, $(x_1, \dots, x_n)$ are the *standard coordinates* on $\mathbb{R}^n$.

Suppose now that $T : V \to W$ is a linear transformation, where W is a vector space with basis $\gamma = (\mathbf{w}_1, \dots, \mathbf{w}_m)$. We can then apply T to equation (1) and obtain

$$T(\mathbf{v}) = \begin{pmatrix} T(\mathbf{v}_1) & T(\mathbf{v}_2) \dots & T(\mathbf{v}_n) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ x_n \end{pmatrix}. \tag{2}$$

For each choice of k, $1 \leq k \leq n$, we $T(\mathbf{v}_k)$ uniquely as a linear combination of $(\mathbf{w}_1, \dots, \mathbf{w}_m)$:

$$T(\mathbf{v}_k) = \begin{pmatrix} \mathbf{w}_1 & \mathbf{w}_2 \dots & \mathbf{w}_m \end{pmatrix} \begin{pmatrix} a_{1k} \\ a_{2k} \\ \cdot \\ a_{mk} \end{pmatrix}.$$

It follows that

$$\begin{pmatrix} T(\mathbf{v}_1) & T(\mathbf{v}_2) & \dots & T(\mathbf{v}_n) \end{pmatrix} = \begin{pmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \dots & \mathbf{w}_m \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

and (2) becomes

$$T(\mathbf{v}) = \begin{pmatrix} \mathbf{w}_1 & \dots & \mathbf{w}_m \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdots & \cdot \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \cdot \\ x_n \end{pmatrix}. \tag{3}$$

We say that

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

is the *matrix of* T with respect to the bases $\beta = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ and $\gamma = (\mathbf{w}_1, \dots, \mathbf{w}_m)$, and write

$$A = \mathcal{M}(T, \beta, \gamma) = \mathcal{M}(T, (\mathbf{v}_1, \dots, \mathbf{v}_n), (\mathbf{w}_1, \dots, \mathbf{w}_m)).$$

We shorten the notation to $A = \mathcal{M}(T)$ when there is no danger of confusion. If

$$T(\mathbf{v}) = \begin{pmatrix} \mathbf{w}_1 & \dots & \mathbf{w}_m \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \cdot \\ y_m \end{pmatrix},$$

then it follows from (3) that

$$\begin{pmatrix} y_1 \\ y_2 \\ \cdot \\ y_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ x_n \end{pmatrix}. \tag{4}$$

So the coordinates of $T(\mathbf{w})$ are related to the coordinates of $\mathbf{v}$ by left multiplication by the matrix A.

Note that if A is an $m \times n$ matrix and T_A is the linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ defined by $T_A(\mathbf{x}) = A\mathbf{x}$ it follows from (4) that with respect to the standard bases $\mathcal{M}(T_A) = A$. The columns of A are the images of the standard basis vectors.

Suppose that we want to represent a counterclockwise rotation of $\mathbb{R}^2$ through an angle θ by means of a linear map T. This linear map must satisfy

$$T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}.$$

Thus with respect to the standard basis,

$$\begin{pmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{pmatrix} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{so} \quad \mathcal{M}(T) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

In terms of the standard coordinates on $\mathbb{R}^2$,

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Note that the basis elements transform by multiplication by $\mathcal{M}(T)$ on the right, while the coordinates transform by multiplication by $\mathcal{M}(T)$ on the left.

Here is another example. Suppose that $\mathcal{P}_2(\mathbb{R})$ denotes the space of polynomials of degree two, with the basis β, which consists of

$$p_0(x) = 1, \quad p_1(x) = x, \quad p_2(x) = x^2,$$

and that $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ is the linear transformation defined by

$$T(p(x)) = p'(x) = \frac{dp}{dx}(x).$$

Here $V = W$ and we choose the same basis in both V and W. In this case,

$$T(1) = 0 = \begin{pmatrix} 1 & x & x^2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad T(x) = 1 = \begin{pmatrix} 1 & x & x^2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$T(x^2) = 2x = \begin{pmatrix} 1 & x & x^2 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix},$$

and hence

$$\mathcal{M}(T, \beta, \beta) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

If instead

$$T(p(x)) = p'(x) - p(x) = \frac{dp}{dx}(x) - p(x),$$

then

$$T(1) = \begin{pmatrix} 1 & x & x^2 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \quad T(x) = \begin{pmatrix} 1 & x & x^2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},$$

$$T(x^2) = \begin{pmatrix} 1 & x & x^2 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix},$$

and hence

$$\mathcal{M}(T, \beta, \beta) = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{pmatrix}.$$