Math 5BI: Problem Set 14
Surface area

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Sometimes surfaces in $\mathbb{R}^3$ are conveniently represented as graphs of functions, sometimes as the images of smooth maps, called parametrizations.

For example, the *paraboloid of revolution* in $\mathbb{R}^3$, defined as the set of points which satisfy the equation

$$z = x^2 + y^2,$$

can be thought of as the graph of the function $f(x, y) = x^2 + y^2$. But we can also regard it as the image of the map

$$\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3 \quad \text{by} \quad \mathbf{x}(u, v) = \begin{pmatrix} u \\ v \\ u^2 + v^2 \end{pmatrix}.$$

The map $\mathbf{x}$ is called a parametrization for the paraboloid of revolution.

A *parametrization* of a smooth surface S is simply a smooth one-to-one map $\mathbf{x}$ from a domain D in the (u, v) plane onto S.

A plane passing through the origin of $\mathbb{R}^3$ is simply a two-dimensional linear subspace—it can be parametrized by the mapping $\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3$ given by the formula

$$\mathbf{x}(u, v) = u\mathbf{b}_1 + v\mathbf{b}_2,$$

where $\mathbf{b}_1, \mathbf{b}_2$ is a basis for the plane. The plane that passes through the point $\mathbf{p}$ and is parallel to the linearly independent vectors $\mathbf{b}_1, \mathbf{b}_2$ is parametrized by $\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3$, where

$$\mathbf{x}(u, v) = \mathbf{p} + u\mathbf{b}_1 + v\mathbf{b}_2.$$

Problem 14.1. a. Show that $\{(1, 0, 1), (0, 1, 1)\}$ is a basis for the linear subspace of $\mathbb{R}^3$ defined as the set of solutions to the homogeneous linear equation $x + y - z = 0$.

b. Find a parametrization for the plane $x + y - z = 7$.

The sphere of radius a centered at the origin, $x^2 + y^2 + z^2 = a^2$, can be parametrized in terms of spherical coordinates:

$$\begin{cases} x = r\cos\theta\sin\phi, \\ y = r\sin\theta\sin\phi, \\ z = r\cos\phi. \end{cases}$$

In these spherical coordinates, the sphere is represented by the equation $r = a$, so we can parametrize the sphere by $\mathbf{x} : D \to \mathbb{R}^3$, where

$$D = \{(\theta, \phi) \in \mathbb{R}^2 : 0 < \theta < 2\pi, 0 < \phi < \pi\}$$

and

$$\mathbf{x}(\theta, \phi) = \begin{pmatrix} a\cos\theta\sin\phi \\ a\sin\theta\sin\phi \\ a\cos\phi \end{pmatrix}.$$

Actually, this parametrization does not cover the entire sphere. It misses the "prime meridian," a subset of zero area, which will not affect our subsequent calculations of surface integrals.

Problem 14.2. Find a parametrization of the ellipsoid,

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1,$$

where a, b, and c are positive, by introducing the new variables

$$u = \frac{x}{a}, v = \frac{y}{b}, w = \frac{z}{c}.$$

so that the equation of the ellipsoid simplifies to $u^2 + v^2 + w^2 = 1$, and then use our previous parametrization of the sphere.

A surface obtained from a smooth curve in the right half of the (x, z)-plane by rotating the curve about the z-axis is called a *surface of revolution*. Surfaces of revolution can be conveniently parametrized by means of cylindrical coordinates. For example, suppose that the curve in the (x, z)-plane is the catenary $x = \cosh z$. In terms of cylindrical coordinates

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \\ z = z. \end{cases}$$

the surface revolution generated by the catenary is represented by the equation $r = \cosh z$. This surface is called the *catenoid*. We can parametrize the catenoid by setting $z = u$ and $\theta = v$.

Problem 14.3. a. Sketch the catenoid.

b. Find a parametrization $\mathbf{x} : D \to \mathbb{R}^3$ of the catenoid, where

$$D = \{(u, v) \in \mathbb{R}^2 : 0 \le v < 2\pi\}.$$

Now we turn to the problem of calculating surface area. We start with the observation that the area of the parallelogram spanned by two vectors $\mathbf{v}$ and $\mathbf{w}$ is simply the length of their cross product,

$$\text{Area} = |\mathbf{v} \times \mathbf{w}|.$$

Suppose now that $\mathbf{x} : D \to \mathbb{R}^3$ is the parametrization of a surface S, and that $(u_0, v_0) \in D$. Let $\Box$ denote the rectangular region in D with corners at (u_0, v_0), $(u_0 + du, v_0)$, $(u_0, v_0 + dv)$, and $(u_0 + du, v_0 + dv)$. The linearization of $\mathbf{x}$ at (u_0, v_0) is the affine mapping

$$\mathbf{L}(\mathbf{x}) = \mathbf{x}(u_0, v_0) + \frac{\partial \mathbf{x}}{\partial u}(u - u_0) + \frac{\partial \mathbf{x}}{\partial v}(v - v_0).$$

Under this affine mapping

$$(u_0, v_0) \mapsto \mathbf{x}(u_0, v_0), \quad (u_0 + du, v_0) \mapsto \mathbf{x}(u_0, v_0) + \frac{\partial \mathbf{x}}{\partial u}du,$$

$$(u_0, v_0 + dv) \mapsto \mathbf{x}(u_0, v_0) + \frac{\partial \mathbf{x}}{\partial v}dv, \quad (u_0 + du, v_0 + dv) \mapsto \mathbf{x}(u_0, v_0) + \frac{\partial \mathbf{x}}{\partial u}du + \frac{\partial \mathbf{x}}{\partial v}dv.$$

The four image points are the corners of the parallelogram located at $\mathbf{x}(u_0, v_0)$ and spanned by

$$\frac{\partial \mathbf{x}}{\partial u}(u_0, v_0)du \quad \text{and} \quad \frac{\partial \mathbf{x}}{\partial v}(u_0, v_0)dv,$$

a parallelogram which has area

$$dA = \left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right| dudv.$$

Since the linearization closely approximates the parametrization $\mathbf{x} : D \to R^3$ near (u_0, v_0), the area of $\mathbf{x}(\Box)$ is closely approximated by

$$dA = \left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right| dudv.$$

If we divide D up into many small rectangles like $\Box$ and add up their contributions to the area, we obtain the following formula for the surface area of a surface S parametrized by $\mathbf{x} : D \to \mathbb{R}^3$:

$$\text{Surface area of } S = \int\int_D \left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right| dudv.$$

For example suppose that we want to find the area of the sphere which is defined by the equation $x^2 + y^2 + z^2 = a^2$. We can use the parametrization $\mathbf{x} : D \to \mathbb{R}^3$, where

$$D = \{(\theta, \phi) \in R^2 : 0 < \theta < 2\pi, 0 < \phi < \pi\}$$

and

$$\mathbf{x}(\theta, \phi) = \begin{pmatrix} a \cos\theta \sin\phi \\ a \sin\theta \sin\phi \\ a \cos\phi \end{pmatrix}.$$

Problem 14.4. Find the surface area of the sphere of radius a.

Problem 14.5. Find the surface area of that part of the paraboloid $z = x^2 + y^2$ which lies inside the cylinder $x^2 + y^2 = 1$.

Problem 14.6. Find the surface area of that part of the cone $x^2 + y^2 = z^2$ which lies between the planes $z = 0$ and $z = 2$.

Problem 14.7. Find the surface area of that part of the catenoid $x^2 + y^2 = \sinh^2 z$ which lies between the planes $z = -1$ and $z = 1$.

Problem 14.8. Let S be the *torus* defined by the equation

$$(\sqrt{x^2 + y^2} - 2)^2 + z^2 = 1,$$

with the parametrization $\mathbf{x} : D \to S$, defined by

$$\mathbf{x}(u, v) = \begin{pmatrix} (2 + \cos v) \cos u \\ (2 + \cos v) \sin u \\ \sin v \end{pmatrix},$$

where

$$D = \{(u, v) \in \mathbb{R}^2 : -\pi < u < \pi, -\pi < v < \pi\}.$$

Find the surface area of S.

If $\mathbf{x} : D \to R^3$ is the parametrization of a surface $\mathbf{S}$ and $f(x, y, z)$ is any continuous function of three variables, the *surface integral* of f over $\mathbf{S}$ is given by the formula

$$\int\int_{\mathbf{S}} f(x, y, z) dA = \int\int_D f(x(u, v), y(u, v), z(u, v)) \left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right| dudv.$$

In more advanced texts it is shown that the integral thus defined is independent of parametrization.

Problem 14.9. If

$$\mathbf{S} = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = a^2, z \geq 0\}.$$

evaluate the surface integral

$$\int\int_{\mathbf{S}} z dA.$$

SUPPLEMENTARY MATERIAL: FLUX INTEGRALS

Suppose now that we have a continuous choice of unit-normal $\mathbf{N}$ to the surface $\mathbf{S}$. Such a continuous choice of unit-normal is called an *orientation* of $\mathbf{S}$. If

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$$

is a smooth vector field on R^3 the *flux* of $\mathbf{F}$ through $\mathbf{S}$ is given by the surface integral

$$\int\int_{\mathbf{S}} \mathbf{F} \cdot \mathbf{N} dA.$$

Calculation of flux integrals is simpler than might be expected, because

$$\mathbf{N}dA = \frac{\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}}{\left|\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\right|} \left|\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\right| dudv = \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} dudv,$$

and hence

$$\int\int_{\mathbf{S}} \mathbf{F} \cdot \mathbf{N}dA = \int\int_{D} \mathbf{F}(x(u,v), y(u,v), z(u,v)) \cdot \left(\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\right) dudv.$$

Problem 14.10. If $\mathbf{S}$ is the hemisphere $x^2 + y^2 + z^2 = a^2, z \geq 0$, and

$$\mathbf{F}(x,y,z) = y\mathbf{i} - x\mathbf{j} + z^3\mathbf{k},$$

evaluate

$$\int\int_{\mathbf{S}} \mathbf{F} \cdot \mathbf{N}dA.$$

A physical picture for the flux integral: Suppose that a fluid is flowing throughout (x,y,z)-space with velocity $\mathbf{V}(x,y,z)$ and density $\rho(x,y,z)$. In this case, fluid flow is represented by the vector field

$$\mathbf{F} = \rho\mathbf{V},$$

and the surface integral

$$\int\int_{\mathbf{S}} \mathbf{F} \cdot \mathbf{N}dA \tag{1}$$

represents the rate at which the fluid is flowing accross $\mathbf{S}$ in the direction of $\mathbf{N}$.

To see this, note first that the rate at which fluid flows across a small piece of $\mathbf{S}$ of surface area dA is

$$(\text{density})(\text{normal component of velocity})dA = \rho\mathbf{V} \cdot \mathbf{N}dA.$$

If we add up the contributions of all the small area elements, we obtain the integral (1).