Math 240B: Problem Set III

February 6, 2009

Due: Friday, February 13, 2009.

Exercise V. We consider the special case of the construction in §1.13 of the lecture notes in which $G = U(n)$ and s is conjugation with

$$I_{1,n-1} = \begin{pmatrix} -1 & 0 \\ 0 & I_{(n-1)\times(n-1)} \end{pmatrix},$$

so that the fixed point set of the automorphism s is $H = U(1) \times U(n-1)$ and $G/H = \mathbb{C}P^{n-1}$.

a. Recall that the Lie algebra $\mathfrak{u}(n)$ divides into a direct sum $\mathfrak{u}(n) = \mathfrak{h} \oplus \mathfrak{p}$, where

$$\mathfrak{h} = \{X \in \mathfrak{g} : s_*(X) = X\}, \quad \mathfrak{p} = \{X \in \mathfrak{g} : s_*(X) = -X\},$$

where $\mathfrak{h}$ is the Lie algebra of $U(1) \times U(n-1)$. Consider two elements

$$X = \begin{pmatrix} 0 & -\bar{\xi}_2 & \cdots & -\bar{\xi}_n \\ \xi_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdot \\ \xi_n & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & -\bar{\eta}_2 & \cdots & -\bar{\eta}_n \\ \eta_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdot \\ \eta_n & 0 & \cdots & 0 \end{pmatrix}$$

of $\mathfrak{p}$, and determine their Lie bracket $[X, Y] \in \mathfrak{h}$.

b. Use the formula for curvature of G/H to show that the sectional curvatures $K(\sigma)$ for $\mathbb{C}P^{n-1}$ satisfy the inequalities $a^2 \leq K(\sigma) \leq 4a^2$ for some $a^2 > 0$.

Remark. The Riemannian metric defined on $G/H = \mathbb{C}P^{n-1}$ is called the *Fubini-Study metric*. It occurs frequently in algebraic geometry.