Math 240B: Problem Set IV

February 13, 2009

Due: Monday, February 23, 2009.

Exercise VI. Show that if X and Y are smooth vector fields on M and $\theta \in \Omega^1(M)$ is a smooth one-form, then

$$d\theta(X,Y) = X(\theta(Y)) - Y(\theta(X)) - \theta([X,Y]).$$

Hint: Use local coordinates.

Exercise VII. For $1 \leq i, j \leq n$, define functions $x^i_j, y^i_j : GL(n,\mathbb{R}) \to \mathbb{R}$ by

$$x^i_j \begin{pmatrix} a^1_1 & \cdots & a^1_n \\ \cdot & \cdots & \cdot \\ a^n_1 & \cdots & a^n_n \end{pmatrix} = a^i_j, \quad y^i_j = x^i_j(A^{-1}).$$

a. Show that the differential form

$$\omega^i_j = \sum_{k=1}^n y^i_k dx^k_j$$

is left invariant.

b. Establish the *Maurer-Cartan equations* for $GL(n,\mathbb{R})$:

$$d\omega^i_j = -\sum_{k=1}^n \omega^i_k \wedge \omega^k_j.$$