Review of Linear Independence Theorems

Math 108 A: Spring 2010
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A. The Linear Dependence Lemma and Replacement Theorem.

Primary goals of this course include learning the notions of linear independence and spanning, and developing the ability to prove theorems from linear algebra that use these concepts. Thus you should be able to reproduce the following definitions:

Definition. Suppose that V is a vector space over a field F. A list of vectors $(\mathbf{v}_1, \ldots \mathbf{v}_n)$ from V is said to *span* V if

$$\mathbf{v} \in V \quad \Rightarrow \quad \mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n,$$

for some choice of $a_1, a_2, \ldots, a_n \in F$.

Definition. Suppose that V is a vector space over a field F. A list of vectors $(\mathbf{v}_1, \ldots \mathbf{v}_n)$ from V is said to be *linearly independent* if

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n = \mathbf{0} \quad \Rightarrow \quad a_1 = a_2 = \cdots = a_n = 0.$$

The list of vectors is said to be *linearly dependent* if it is not linearly independent.

You should know how to prove:

Linear Dependence Lemma. *Suppose that $(\mathbf{v}_1, \ldots, \mathbf{v}_m)$ is a linearly dependent list of vectors in a vector space V over a field F, and that $\mathbf{v}_1 \neq \mathbf{0}$. Then there exists $j \in \{2, \ldots, m\}$ such that*

$$\mathbf{v}_j \in \mathit{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}).$$

Moreover,

$$\mathit{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots \mathbf{v}_m) = \mathit{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_m).$$

You should know how to prove this Lemma (as worked out in the online key to Quiz C). Assuming this lemma, we prove the MAIN RESULT of Chapter 1 in the text:

Replacement Theorem. *If V is a vector space over a field F, $(\mathbf{u}_1, \ldots, \mathbf{u}_m)$ is a linearly independent list of elements from V, and V is the span of a list $(\mathbf{w}_1, \ldots, \mathbf{w}_n)$, then $m \leq n$.*

Idea of proof: One by one replace elements of the spanning list by elements of the linear independent list, renormalizing to the same size by means of the Linear Dependence Lemma.

You should know how to carry out the details of the proof of the Replacement Theorem (as worked out in the key to Quiz D).

Definition. A *basis* for a vector space V is a list $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ which is linearly independent and spans V.

Corollary of Replacement Theorem. *If V is a vector space over a field F, Any two finite bases for V have the same number of elements.*

The proof is written out in Practice Quiz D.

Definition. A vector space V over a field F is *finite-dimensional* if it has a basis which has finitely many elements. If it is not finite-dimensional it is said to be *infinite-dimensional*. The *dimension* of a finite-dimensional vector space V is the number of elements in any of its bases. We let $\dim(V)$ denote the dimension of V.

B. Consequences.

The Linear Dependence Lemma and the Replacement Theorem can be used to prove the most difficult of the other theorems on linear dependence and independence. For example, here are two proofs that you should know how to present:

You should know how to use the Linear Dependence Lemma to prove the

Reduction Theorem. *Every spanning list in a vector space V can be reduced to a basis.*

Idea: The idea of the proof is to start with a spanning list and throw away elements until you have a basis. As long as you don't have a basis, the Linear Dependence Lemma says that you can throw something away.

Proof: Suppose that $V = \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$. Start with the list $L = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$.

For each j, $1 \leq j \leq n$ ask whether $\mathbf{v}_j$ is in the span of the previous elements of the list. If so, throw it away, obtaining a new list. If not, keep $\mathbf{v}_j$ in the list.

Repeat this procedure n times obtaining a new list L which spans V. The Linear Dependence Lemma now implies that this list is linearly independent. (If not, one of the elements would have been in the span of the previous elements and would have been thrown away.) Hence L is a basis for V.

You should also know how to use the Linear Dependence Lemma to prove the following:

Extension Theorem. *Every linearly independent list in a finite-dimensional vector space V can be extended to a basis.*

Idea: The idea of the proof is to suppose that $(\mathbf{u}_1, \ldots, \mathbf{u}_m)$ is a linearly independent list in V. Since V is finite-dimensional, we can write $V = \text{span}(\mathbf{w}_1, \ldots, \mathbf{w}_n)$. One by one, add the $\mathbf{w}_i$'s to the list L, throwing away any additions that make the list linearly dependent (using the Linear Dependence Lemma).

Proof: Suppose that $L = (\mathbf{u}_1, \ldots, \mathbf{u}_m)$ is a linearly independent list in V. Since V is finite-dimensional, we can write $V = \text{span}(\mathbf{w}_1, \ldots, \mathbf{w}_n)$.

Step 1. If $\mathbf{w}_1$ is in the span of L, throw it away. Otherwise, add $\mathbf{w}_1$ to the end of the list. $\mathbf{w}_1$ is in the span of the resulting list and the Linear Dependence Lemma shows that the list is linearly independent.

Step k. Suppose that we have constructed a linearly independent list L extending $(\mathbf{u}_1, \ldots, \mathbf{u}_m)$ such that $\mathbf{w}_1, \ldots, \mathbf{w}_{k-1}$ are in the span of L. If $\mathbf{w}_k$ is in the span of L, throw it away. Otherwise, add $\mathbf{w}_k$ to the list. $\mathbf{w}_1, \ldots, \mathbf{w}_k$ are in the span of the new list and the list is linearly independent by the Linear Dependence Lemma.

After n steps, one obtains a list which extends $(\mathbf{u}_1, \ldots, \mathbf{u}_m)$, spans V and is linear independent. The resulting list is a basis which extends $(\mathbf{u}_1, \ldots, \mathbf{u}_m)$.

C. Linear transformations.

Suppose that V and W are vector spaces over a field F and that $T : V \to W$ is a linear transformation. You should definitely be able to reproduce the following definitions:

Definition. The *null space* of a linear transformation T is

$$N(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\}.$$

Definition. The *range* of a linear transformation T is

$$R(T) = \{\mathbf{w} \in W : \mathbf{w} = T(\mathbf{v}) \text{ for some } \mathbf{v} \in V\ \}.$$

The Main Theorems from Chapter 2 of the text is:

Linear Transformation Theorem. *If V is a finite dimensional vector space and $T : V \to W$ is a linear map into a vector space W, then*

$$\dim V = \dim N(T) + \dim R(T).$$

You should know how to prove the Linear Transformation Theorem; a proof is presented in Practice Quiz F.