Math 117: Infinite Sequences

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The three main theorems in the theory of infinite sequences are the Monotone Convergence Theorem, the Cauchy Sequence Theorem and the Subsequence Theorem, all described below.

1 Convergence of infinite sequences

A *sequence* of real numbers is simply a function $s : \mathbb{N} \to \mathbb{R}$. We let s_n denote the value of the function s at $n \in \mathbb{N}$ and sometimes write (s_n) for the sequence. The key definition in the subject is:

Definition. A sequence (s_n) of real numbers is said to *converge* to a real number s if for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$n > N \quad \Rightarrow \quad |s_n - s| < \epsilon.$$

In this case, we write $s = \lim s_n$. A sequence (s_n) of real numbers which does not converge to a real number is said to *diverge*.

For example, the sequence (s_n) defined by $s_n = 1/n$ converges to 0, because given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $1/N < \epsilon$ by the Archimedean property of the real numbers, and

$$n > N \quad \Rightarrow \quad 0 < \frac{1}{n} < \frac{1}{N} \quad \Rightarrow \quad |s_n - 0| = \left| \frac{1}{n} - 0 \right| < \epsilon.$$

On the other hand, the sequence (s_n) defined by $s_n = 1 + (-1)^n$ diverges. Indeed, suppose that this sequence s_n converges to s. Then we can take $\epsilon = 1$, and there exists $N \in \mathbb{N}$ such that

$$n > N \quad \Rightarrow \quad |s_n - s| < 1.$$

If $n > N$ and n is even, then $s_n = 2$ and $s_{n+1} = 0$. Hence,

$$2 = |s_n - s_{n+1}| \leq |s_n - s| + |s - s_{n+1}| < 1 + 1 = 2,$$

a contradiction.

Proposition. *Suppose that* (s_n) *is a sequence of real numbers which converges to* $s \in \mathbb{R}$*. Then* (s_n) *is bounded.*

Proof: Choose $N \in \mathbb{N}$ such that

$$n > N \quad \Rightarrow \quad |s_n - s| < 1,$$

and let

$$M = \sup\{|s_1|, |s_2|, \dots, |s_N|, |s|\} + 1.$$

If $n \leq N$, then $s_n| \leq M$, while if $n \geq N + 1|$,

$$|s_n| \leq |s_n - s| + |s| \leq 1 + |s| \leq M.$$

Thus $|s_n| < M$ for all $n \in \mathbb{N}$, and (s_n) is bounded.

Proposition. *Suppose that* (s_n) *and* (t_n) *are convergent sequences of real numbers with* $\lim s_n = s$ *and* $\lim t_n = t$*. Then*

1. $\lim(s_n + t_n) = s + t$,
2. $\lim(s_n t_n) = st$,
3. $\lim(s_n/t_n) = s/t$, *provided* $t_n \neq 0$ *for all* n *and* $t \neq 0$,

We give a proof of the first of these assertions; proofs of the others can be found in the text [1]; see Theorem 17.1.

Let $\epsilon > 0$ be given. Since (s_n) converges to s, there exists an $N_1 \in \mathbb{N}$ such that

$$n \in \mathbb{N} \quad \text{and} \quad n > N_1 \quad \Rightarrow \quad |s_n - s| < \frac{\epsilon}{2}.$$

Since (t_n) converges to s, there exists an $N_2 \in \mathbb{N}$ such that

$$n \in \mathbb{N} \quad \text{and} \quad n > N_2 \quad \Rightarrow \quad |t_n - t| < \frac{\epsilon}{2}.$$

Let $N = \max(N_1, N_2)$. Then using the triangle inequality, we conclude that

$$n \in \mathbb{N} \quad \text{and} \quad n > N \quad \Rightarrow \quad |(s_n + t_n) - (s + t)| \leq |s_n - s| + |t_n - t| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which is exactly what we needed to prove.

Application. We can use this theorem, for example, to find the limit

$$\lim \frac{2n+1}{n+2} = \lim \frac{2 + (1/n)}{1 + 2/n} = \frac{\lim(2 + (1/n)}{\lim(1 + 2/n)} = \frac{2 + \lim(1/n)}{1 + \lim(2/n)} = 2.$$

It is useful to have criteria which guarantee convergence of sequences. One of these concerns increasing and decreasing sequences.

A sequence (s_n) of real numbers is said to be *increasing* if $s_n \le s_{n+1}$ for all n. It is *decreasing* if $s_n \ge s_{n+1}$ for all n. A sequence (s_n) of real numbers is *monotone* if it is either increasing or decreasing.

Monotone Convergence Theorem. *A bounded monotone sequence (s_n) of real numbers must converge.*

Proof: We first consider the case in which (s_n) is increasing. By hypothesis, $S = \{s_n : n \in \mathbb{N}\}$ is bounded. By the completeness axiom, it must therefore have a supremum s. Let $\epsilon > 0$. Since $s = \sup S$, there exists $N \in \mathbb{N}$ such that $s_N > s - \epsilon$. Then

$$n > N \quad \Rightarrow \quad s_n \ge s_N > s - \epsilon \quad \text{and} \quad s_n \le s \quad \Rightarrow \quad |s_n - s| < \epsilon.$$

Hence $s = \lim s_n$.

The case where (s_n) is decreasing is proven in a similar fashion.

Definition. A sequence (s_n) of real numbers is said to *diverge* to ∞ (and we write $\lim s_n = \infty$) if for every $M \in \mathbb{R}$, there is an $N \in \mathbb{N}$ such that

$$n > N \quad \Rightarrow \quad s_n > M.$$

A sequence (s_n) of real numbers is said to *diverge* to $-\infty$ (and we write $\lim s_n = -\infty$) if for every $M \in \mathbb{R}$, there is an $N \in \mathbb{N}$ such that

$$n > N \quad \Rightarrow \quad s_n < M.$$

It is not difficult to show that if an increasing sequence (s_n) is not bounded, the sequence diverges to ∞. Similarly, if a decreasing sequence (s_n) is not bounded, the sequence diverges to $-\infty$.

2 Cauchy sequences

It is often useful to consider sequences from more general spaces than just $\mathbb{R}$. A sequence of elements from a metric space (X, d) is just a function from $\mathbb{N}$ to X; we will denote such a sequence by (x_n).

Definition. A sequence (x_n) of elements in a metric space (X, d) is said to *converge* to an element $x \in X$ if for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$n > N \quad \Rightarrow \quad d(x_n, x) < \epsilon.$$

A sequence (x_n) of elements in (X, d) is called a *Cauchy sequence* if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$n, m \in \mathbb{N} \quad \text{and} \quad n, m > N \quad \Rightarrow \quad d(x_n, x_m) < \epsilon.$$

For us, the most important cases are the cases in which $X = \mathbb{R}$ or $X = \mathbb{R}^n$. In the case of $\mathbb{R}^n$,

$$d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|.$$

Cauchy Theorem. *A Cauchy sequence in $\mathbb{R}^n$ must converge.*

Note. It is not true that a Cauchy sequence in an arbitrary metric space is convergent. It is true in $\mathbb{R}$ or in $\mathbb{R}^n$ with the standard metric, because the Heine-Borel theorem and the Bolzano-Weierstrass Theorem are available.

Sketch of Proof: Suppose that $(\mathbf{x}_n)$ is a Cauchy sequence in $\mathbb{R}^n$. We divide into two cases.

Case I: $S = \{\mathbf{x}_n : n \in \mathbb{N}\}$ **is finite.** In this case, we can set

$$\epsilon = \min\left\{|\mathbf{x}_n - \mathbf{x}_m)| : \mathbf{x}_n \neq \mathbf{x}_m,\ n, m \in \mathbb{N}\right\},$$

a positive number, since there are only finitely many distances between distinct elements of S. By definition of Cauchy sequence, there exists an element $N \in \mathbb{N}$ such that

$$n, m \in \mathbb{N} \quad \text{and} \quad n, m > N \quad \Rightarrow \quad |\mathbf{x}_n - \mathbf{x}_m| < \epsilon.$$

But this implies that $\mathbf{x}_n = \mathbf{x}_m$ for $n, m > N$, and hence the Cauchy sequence converges.

Case II: $S = \{\mathbf{x}_n : n \in \mathbb{N}\}$ **is infinite.** In this case, we first show that S is bounded. To do this, we first use the fact that $(\mathbf{x}_n)$ is a Cauchy sequence to choose $N \in \mathbb{N}$ such that

$$n, m \in \mathbb{N} \quad \text{and} \quad n, m > N \quad \Rightarrow \quad |\mathbf{x}_n - \mathbf{x}_m| < 1.$$

We then let

$$M = \sup\{|\mathbf{x}_1|, |\mathbf{x}_2|, \ldots, |\mathbf{x}_N|, |\mathbf{x}_{N+1}|\} + 1.$$

If $n \leq N$, then $|\mathbf{x}_n| \leq M$, while if $n \geq N + 1|$,

$$|\mathbf{x}_n| \leq |\mathbf{x}_n - \mathbf{x}_{N+1}| + |\mathbf{x}_{N+1}| \leq 1 + |\mathbf{x}_{N+1}| \leq M.$$

Thus $|\mathbf{x}_n| < M$ for all $n \in \mathbb{N}$.

Continuing with case II, we note that since $(\mathbf{x}_n)$ is a Cauchy sequence, there exists $N \in \mathbb{N}$ such that

$$n, m > N \quad \Rightarrow \quad |\mathbf{x}_n - \mathbf{x}_m| < \epsilon/2.$$

It follows from the Bolzano-Weierstrass Theorem that S has an accumulation point $\mathbf{x}$. Since $\mathbf{x}$ is an accumulation point, infinitely many points $(\mathbf{x}_n)$ must lie in $N(\mathbf{x}; \epsilon/2)$. Thus there exists an $n > N$ such that $\mathbf{x}_n \in N(\mathbf{x}; \epsilon/2)$. Then

$$m \in \mathbb{N} \quad \text{and} \quad m > N \quad \Rightarrow \quad |\mathbf{x}_m - \mathbf{x}| < |\mathbf{x}_m - \mathbf{x}_n| + |\mathbf{x}_n - \mathbf{x}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $(\mathbf{x}_n)$ converges to $\mathbf{x}$, and hence $(\mathbf{x}_n)$ is bounded.

We say that a metric space (X, d) is *complete* if every Cauchy sequence in X converges. Thus the above theorem states that $(\mathbb{R}^n, d)$ is complete when d is the standard metric on $\mathbb{R}^n$.

3 Subsequences

Suppose that (s_n) is a sequence of real numbers. If $n_1 < n_2 < \cdots < n_k < \dots$ is an increasing sequence of natural numbers, then (s_{n_k}) is called a *subsequence* of (s_n). Even though a bounded sequence of real numbers need not converge, we have:

Subsequence Theorem. *A bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.*

As in the Cauchy theorem, the proof rests on the Bolzano-Weierstrass Theorem. Suppose that $(\mathbf{x}_n)$ is a bounded sequence in $\mathbb{R}^n$. We divide into two cases.

Case I: $S = \{\mathbf{x}_n : n \in \mathbb{N}\}$ is finite. In this case, $\mathbf{x}_n$ is a fixed element $\mathbf{x}_0$ of $\mathbb{R}^n$ for infinitely many $n \in \mathbb{N}$. We can let n_1 be the smallest element of $\mathbb{N}$ such that $\mathbf{x}_{n_1} = \mathbf{x}_0$, n_2 be the second smallest element of $\mathbb{N}$ such that $\mathbf{x}_{n_2} = \mathbf{x}_0$, and so forth. We thereby obtain a subsequence $(\mathbf{x}_{n_k})$ of $(\mathbf{x}_n)$ which converges to $\mathbf{x}_0$.

Case II: $S = \{\mathbf{x}_n : n \in \mathbb{N}\}$ is infinite. Since S is bounded, it follows from the Bolzano-Weierstrass Theorem that S has an accumulation point $\mathbf{x}$.

We choose n_1 so that $\mathbf{x}_{n_1} \in N^*(\mathbf{x}; 1)$ and let

$$\epsilon_1 = \min\left(|\mathbf{x}_{n_1} - \mathbf{x}|, \frac{1}{2}\right).$$

We next choose n_2 so that $\mathbf{x}_{n_2} \in N^*(\mathbf{x}; \epsilon_1)$ and let

$$\epsilon_2 = \min\left(|\mathbf{x}_{n_2} - \mathbf{x}|, \left(\frac{1}{2}\right)^2\right).$$

Continuing in this fashion, we obtain a subsequence $(\mathbf{x}_{n_k})$ of $(\mathbf{x}_n)$ such that

$$|(\mathbf{x}_{n_k} - \mathbf{x}| < \left(\frac{1}{2}\right)^k.$$

The subsequence $(\mathbf{x}_{n_k})$ converges to $\mathbf{x}$.

Definition. Let (s_n) be a sequence of real numbers, and let

$$M_n = \sup\{s_n, s_{n+1}, s_{n+2}, \dots\}, \quad m_n = \inf\{s_n, s_{n+1}, s_{n+2}, \dots\}.$$

Then (M_n) is a monotone decreasing sequence, while (m_n) is a monotone increasing sequence. We let

$$\limsup(s_n) = \lim M_n, \qquad \liminf(s_n) = \lim m_n.$$

Note that $\liminf(s_n) \leq \limsup(s_n)$, with equality holding if and only if (s_n) converges.

Proposition. Suppose that (s_n) is a sequence of real numbers and let

$$S = \{ \text{ all limits of subsequences of } (s_n) \ \}.$$

Then $\sup S = \limsup(s_n)$ and $\inf S = \liminf(s_n)$.

The proof is beyond the scope of the course; we refer to [1], §19 for many examples.

4 Infinite series

Definition. An *infinite series* is a sum of the form

$$\sum_{n=0}^{\infty} a_n,$$

where the a_n's are real numbers. The infinite series is said to *converge* to a real number s if the partial sum

$$s_n = \sum_{m=0}^{n} a_m$$

converges to s. In this case, we write

$$s = \sum_{n=0}^{\infty} a_n.$$

An infinite series which does not converge to a real number is said to *diverge*.

Example. One of the most important infinite series is the *geometric series*

$$\sum_{n=0}^{\infty} x^n, \quad \text{where} \quad |x| < 1.$$

In this case, we have the partial sum

$$s_n = \sum_{m=0}^{n} x^m = 1 + x + \cdots + x^n.$$

Since

$$xs_n = x + x^2 + \cdots + x^{n+1},$$

we find that

$$s_n - xs_n = 1 - x^{n+1}, \quad \text{or} \quad s_n = \frac{1 - x^{n+1}}{1 - x}.$$

Under the assumption that $0 < x < 1$, $\lim x^{n+1} = 0$, from which one can conclude that

$$\lim s_n = \frac{1}{1-x} \quad \text{or} \quad \sum_{m=0}^{\infty} x^m = \frac{1}{1-x}.$$

It is customary to define the real number e by means of the infinite series,

$$\sum_{n=0}^{\infty} \frac{1}{n!}.$$

To see that this infinite series converges, one can compare its partial sums with those of a geometric series,

$$s_n = 1 + 1 + \frac{1}{2} + \frac{1}{3 \cdot 2} + \cdots + \frac{1}{n!} \leq 1 + 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^n \leq 3.$$

We see that the increasing sequence (s_n) is bounded by the sum of the geometric series. It therefore follows from the Monotone Convergence Theorem that (s_n) is a convergent series. We are therefore justified in letting e denote the limit,

$$e = \sum_{m=0}^{\infty} \frac{1}{m!}.$$

Theorem. $\lim \left(1 + (1/n)\right)^n = e$.

Our proof follows Rudin [2], 3.31. We let

$$s_n = \sum_{m=0}^{n} \frac{1}{m!} \quad \text{and} \quad t_n = \left(1 + \frac{1}{n}\right)^n.$$

Then by the binomial theorem

$$\begin{aligned} t_n &= 1 + n\frac{1}{n} + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3 + \cdots \\ &= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \\ &\qquad + \cdots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right). \end{aligned}$$

It follows $t_n \leq t_{n+1}$, so that (t_N) is an increasing sequence, and it also follows from the above expressions that $t_n \leq s_n$. Hence (t_n) is a bounded increasing sequence and $\lim t_n$ exists and $\lim t_n \leq e$.

On the other hand, if $n \geq m$,

$$\begin{aligned} t_n \geq 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \\ + \cdots + \frac{1}{m!}\left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{m-1}{n}\right). \end{aligned}$$

If we fix m and let $n \to \infty$, we obtain

$$\lim t_n \geq 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{m!}.$$

Letting $m \to \infty$ now yields $\lim t_n \geq e$. Thus t_n converges and its limit is e.

More generally, if $x \in \mathbb{R}$, we can define e^x by means of the infinite series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{1}$$

To see that this infinite series converges, we find it useful to make use of the following:

Proposition. *Suppose that* $|a_n| \leq b_n$, *where* $b_n > 0$, *for each* $n \in \mathbb{N}$. *If* $\sum_{n=0}^{\infty} b_n$ *converges, then* $\sum_{n=0}^{\infty} a_n$ *converges.*

Proof: To prove this, suppose that $\sum_{n=0}^{\infty} b_n$ converges, and let $\epsilon > 0$ be given. Then

$$\exists N \in \mathbb{N} \quad \text{such that} \quad n > m > N \quad \Rightarrow \quad \sum_{k=m}^{n} b_k < \epsilon.$$

It follows that when $n > m > N$,

$$\left| \sum_{k=m}^{n} a_k \right| \leq \sum_{k=m}^{n} |a_k| \leq \sum_{k=m}^{n} b_k < \epsilon.$$

Thus if

$$s_n = \sum_{k=0}^{n} a_k, \quad n > m > N \quad \Rightarrow \quad |s_n - s_m| < \epsilon,$$

which implies that (s_n) is a Cauchy sequence. By the Cauchy Theorem, (s_n) converges and hence $\sum_{n=0}^{\infty} a_n$ converges.

We can apply this to show that this series (1) converges for $|x| < M$, where $M \in \mathbb{N}$ is given. Indeed, we can write

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{M-1} \frac{x^n}{n!} + x^M \sum_{k=0}^{\infty} a_k, \quad \text{where} \quad a_k = \frac{x^k}{(M+k)!}.$$

We can compare the last sum on the right with

$$\sum_{k=0}^{\infty} c_k, \quad \text{where} \quad c_k = \frac{1}{(M)!} \left(\frac{|x|}{M} \right)^k,$$

a constant multiple of a convergent geometric series with positive terms. The preceding proposition shows that the series (1) converges.

By modifying the proof of the preceding theorem, one could now establish the following limit:

$$\lim \left(1 + \frac{x}{n} \right)^n = e^x, \quad \text{for} \quad x \geq 0.$$

5 The Contraction Mapping Theorem

We now discuss a slightly more advanced topic. A metric space (X, d) is said to be *complete* if every Cauchy sequence in (X, d) converges. Thus, for example, it follows from the Cauchy Sequence Theorem that $\mathbb{R}$ and $\mathbb{R}^n$ are complete.

A function $f : X \to X$ is called a *contraction* if for all $x, y \in X$,

$$d(f(x), f(y)) < \alpha d(x, y), \qquad \text{where} \qquad 0 < \alpha < 1. \tag{2}$$

A point $x \in X$ is said to be a *fixed point* of the contraction if $f(x) = x$.

Contraction Mapping Theorem. *If (X, d) is a complete metric space and $f : X \to X$ is a contraction, then f has a unique fixed point.*

Sketch of proof: We start by picking a point $x_0 \in X$, and for $n \in N$, let $x_n = f(x_{n-1})$. This gives a sequence of points (x_n) in X. Suppose that $d(x_0, x_1) = \beta$. Then it follows from (2) that

$$d(x_1, x_2) < \alpha\beta, \quad d(x_2, x_3) < \alpha^2\beta, \quad \ldots, \quad d(x_n, x_{n+1}) < \alpha^n\beta, \quad \ldots.$$

Hence if $k > 0$,

$$\begin{aligned} d(x_n, x_{n+k}) &\leq d(x_n, x_{n+1}) + \cdots d(x_{n+k-1}, x_{n+k}) \\ &< \alpha^n\beta(1 + \alpha + \cdots + \alpha^{k-1}) < \alpha^n \frac{\beta}{1-\alpha}, \end{aligned}$$

where we have used the formula for the sum of a geometric series. Given $\epsilon > 0$, we can choose N sufficiently large that when $n > N$,

$$\alpha^n < \epsilon \frac{1-\alpha}{\beta} \quad \text{and hence} \quad \alpha^n \frac{\beta}{1-\alpha} < \epsilon.$$

Hence for $n, m > N$, $d(x_n, x_m) < \epsilon$ and (x_n) is a Cauchy sequence.

Since (X, d) is complete, the Cauchy sequence (x_n) converges to an element $x \in X$. One shows that this x is a fixed point of f.

If x and y are two fixed points of f, then it follows from (2) that

$$d(x, y) = d(f(x), f(y)) < \alpha d(x, y) \quad \Rightarrow \quad x = y,$$

so the fixed point is unique.

References

[1] Steven R. Lay, *Analysis: with an introduction to proof*, Pearson Prentice Hall, Upper Saddle Riven, NJ, 2005.

[2] W. Rudin, *Principles of mathematical analysis*, Third edition, McGraw-Hill, New York, 1976.