Linear Systems: Diagonalizable Systems

Ideas: We know how to solve the one-dimensional linear system $\dot{x} = \lambda x$—$x(t) = x(0)e^{\lambda t}$. So, if (potentially after a change of variables), we have a system of equations of the form $\dot{x}_j = \lambda_j x_j$, we actually have a solution formula for the system. This can be viewed as a map $\Phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ sending $(t, x(0)) \mapsto x(t)$.

Solve the system

$$\dot{x} = \begin{bmatrix} 7 & -2 \\ 16 & -5 \end{bmatrix} x$$

in terms of Φ. Sketch the phase portrait.

To diagonalize, we compute the characteristic polynomial

$$\det(A - \lambda I) = (7 - \lambda)(-5 - \lambda) - (-2)(16) = \lambda^2 - 2\lambda - 3 = (\lambda + 1)(\lambda - 3).$$

An eigenvector for -1 is in the kernel of $\begin{bmatrix} 8 & -2 \\ 16 & -4 \end{bmatrix}$, so one choice is $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$. An eigenvector for 3 is in the kernel of $\begin{bmatrix} 4 & -2 \\ 16 & -8 \end{bmatrix}$, so one choice is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Thus, by making the change of variables from $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ to $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -1 & \frac{1}{2} \\ 2 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, we have the diagonal system

$$\dot{y} = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} y,$$

giving the solution $y(t) = \begin{bmatrix} e^{-t} y_1(0) \\ e^{3t} y_2(0) \end{bmatrix}$. Changing back to the original coordinates, we have

$$\Phi(t, x_0) = \begin{bmatrix} 1 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} -1 & \frac{1}{2} \\ 2 & -\frac{1}{2} \end{bmatrix} x_0 = \begin{bmatrix} 2e^{3t} - e^{-t} & \frac{1}{2}e^{-t} - \frac{1}{2}e^{3t} \\ 4e^{3t} - 4e^{-t} & 2e^{-t} - e^{3t} \end{bmatrix} x_0.$$

Solve the equation $$\ddot{u} + 4\dot{u} + 3u = 0$$ in terms of $u(0) = u_0$ and $\dot{u}(0) = \dot{u}_0$.

We first set the vector $x = \begin{bmatrix} u \\ \dot{u} \end{bmatrix}$, yielding the system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} x.$$

This has characteristic polynomial $\lambda^2 + 4\lambda + 3 = (\lambda + 1)(\lambda + 3)$. For -1 we have the eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and for -3 we have the eigenvector $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$, so we have the diagonal system

$$\dot{x} = \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} x.$$

Thus, our dynamic map is

$$\Phi(t, x_0) = \frac{1}{2} \begin{bmatrix} 3e^{-t} - e^{-3t} & e^{-t} - e^{-3t} \\ 3e^{-3t} - 3e^{-t} & 3e^{-3t} - e^{-t} \end{bmatrix} x_0.$$

The first row of this gives the function u, and as $x_0 = \begin{bmatrix} u_0 \\ \dot{u}_0 \end{bmatrix}$, we see that

$$u(t) = \left(\frac{3u_0 + \dot{u}_0}{2} \right) e^{-t} - \left(\frac{u_0 + \dot{u}_0}{2} \right) e^{-3t} = \left(\frac{3e^{-t} - e^{-3t}}{2} \right) u_0 + \left(\frac{e^{-t} - e^{-3t}}{2} \right) \dot{u}_0.$$

Suppose we have the linear system

$$\dot{x} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} x.$$

Write down a solution given initial data $x(0) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

We can see the matrix has a zero eigenvalue, as it has a repeated row, and it is not hard to see that the vector $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is in the kernel. Similarly, it is not hard to see that $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are eigenvectors with eigenvalue 2. So, we have a diagonalization

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This implies a solution

$$\begin{aligned} x(t) &= \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \\ &= \begin{bmatrix} 1 & e^{2t} & 0 \\ -1 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} \frac{a-b}{2} \\ \frac{a+b}{2} \\ c \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} a(e^{2t}+1) + \frac{1}{2} b(e^{2t}-1) \\ \frac{1}{2} a(e^{2t}-1) + \frac{1}{2} b(e^{2t}+1) \\ ce^{2t} \end{bmatrix}. \end{aligned}$$