Linear Systems: Exponential Solutions

Ideas: Taking a slightly different approach to diagonalization, if we have a (vector) equation $\dot{x} = Ax$, we can define the *matrix exponential*

$$e^{At} := \lim_{N\to\infty} \sum_{k=1}^{N} \frac{A^k t^k}{k!},$$

where the limit is taken with respect to the *operator norm*. Then our solution is $x(t) = e^{At}x(0)$ (this of course requires proof). Unfortunately, often the matrix exponential is difficult to compute explicitly; a helpful result is that if $ST = TS$, the usual exponential rule $e^{S+T} = e^S e^T$ applies. In particular, if $A = PDP^{-1}$, then $e^A = Pe^D P^{-1}$, so this covers the diagonalizable case.

What is e^{At} when

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}?$$

Here, we can diagonalize A: writing the characteristic polynomial yields

$$\lambda^2 - 2\lambda = \lambda(\lambda - 2),$$

and we can see an eigenvector for 0 is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and one for 2 is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So, we get

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

Taking the exponential then gives

$$e^{At} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^{2t} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{2t}+1 & e^{2t}-1 \\ e^{2t}-1 & e^{2t}+1 \end{bmatrix}.$$

We could also realize that

$$A^n = \begin{bmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{bmatrix},$$

so we have directly

$$e^{At} = I + \frac{1}{2}\left((2t) + \frac{1}{2}(2t)^2 + \frac{1}{6}(2t)^3 + \cdots\right)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{2t}+1 & e^{2t}-1 \\ e^{2t}-1 & e^{2t}+1 \end{bmatrix}.$$

What is the matrix exponential of

$$A = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}?$$

We write $A = \lambda I + N$, where I is the identity matrix and

$$N = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now, $IN = N = NI$, so $e^{\lambda I + N} = e^{\lambda I} e^N = e^\lambda e^N$ (since $I^n = I$ for all $n \in \mathbb{N}$). We can directly compute that $N^2 = 0$, so $e^N = I + N$. Thus, we have

$$e^A = \begin{bmatrix} e^\lambda & e^\lambda & 0 & 0 \\ 0 & e^\lambda & 0 & 0 \\ 0 & 0 & e^\lambda & e^\lambda \\ 0 & 0 & 0 & e^\lambda \end{bmatrix}.$$

As motivation for this decomposition, we consider that N clearly (why?) has zero as its only eigenvalue, so some power of it must be the zero matrix (we call N a *nilpotent* matrix for this reason). Thus, we would like to reduce the computation to an exponential involving this matrix.

What is e^{At} where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}?$$

From geometric intuition, this is a rotation matrix through an angle of $\frac{\pi}{2}$, so its powers should form a cycle. We compute $A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and $A^4 = I$. From here, the sequence repeats ($A^5 = A$, $A^6 = A^2$, ...), so we see that

$$e^{At} = \begin{bmatrix} 1 - \frac{1}{2}t^2 + \frac{1}{24}t^4 - \cdots & t - \frac{1}{6}t^3 + \frac{1}{120}t^5 - \cdots \\ -t + \frac{1}{6}t^3 - \frac{1}{120}t^5 + \cdots & 1 - \frac{1}{2}t^2 + \frac{1}{24}t^4 - \cdots \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}.$$

Suppose we have the linear system

$$\begin{cases} \ddot{x} + \ddot{y} + 3\dot{x} + 3\dot{y} + 2x + 2y = 0 \\ \ddot{x} - \ddot{y} + 5\dot{x} - 5\dot{y} + 6x - 6y = 0. \end{cases}$$

(a) Make a change of variables to decouple the system to two independent second order equations. [Note: this is not always possible, but will greatly simplify the work to come when it is.]

(b) Introduce new variables to find an equivalent first order system of the fourth order.

(c) Diagonalize the system (it should be diagonalizable; to save work, consider that the two sets of variables can be thought of separately since they are independent).

(d) Write down the solution to the system, and re-interpret in terms of the original variables (feel free to leave it as a product of matrices, though).

(a) A good choice of variables is $u = x+y$ and $v = x-y$—this should be suggested by the structure of the equations (x and y enter both with the same coefficients, and the same sign in the upper and opposite signs in the lower equation). Then $x = \frac{1}{2}(u+v)$ and $y = \frac{1}{2}(u-v)$, and we have the equations

$$\begin{cases} \ddot{u} + 3\dot{u} + 2u = 0 \\ \ddot{v} + 5\dot{v} + 6v = 0. \end{cases}$$

(b) We set $z = \dot{u}$ and $w = \dot{v}$ to obtain the matrix system

$$\begin{bmatrix} u \\ z \\ v \\ w \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -6 & -5 \end{bmatrix} \begin{bmatrix} u \\ z \\ v \\ w \end{bmatrix}.$$

This is block diagonal, and we can see the characteristic equations for the blocks are the respective characteristic equations for the two second order equations, so we get

$$(\lambda^2 + 3\lambda + 2)(\lambda^2 + 5\lambda + 6) = (\lambda+1)(\lambda+2)^2(\lambda+3) = 0$$

as our characteristic equation. For the upper block with eigenvalues -1 and -2, we choose eigenvectors $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix}$, respectively. For the lower block with eigenvalues -2 and -3, we choose eigenvectors $\begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ -1 \\ 3 \end{bmatrix}$, respectively. Then we get

$$\begin{bmatrix} u \\ z \\ v \\ w \end{bmatrix}' = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} u \\ z \\ v \\ w \end{bmatrix}$$

as our diagonalization (you can check the inversion yourself, but note that this choice of eigenvectors makes both the upper left and lower right blocks have determinant 1, simplifying the computation).

(c) As this is a diagonalized system, we get a solution

$$\begin{bmatrix} u(t) \\ z(t) \\ v(t) \\ w(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 & 0 & 0 \\ 0 & e^{-2t} & 0 & 0 \\ 0 & 0 & e^{-2t} & 0 \\ 0 & 0 & 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} u(0) \\ z(0) \\ v(0) \\ w(0) \end{bmatrix}.$$

We can return this to our original variables by noting that $z = \dot{x} + \dot{y}$ and $w = \dot{x} - \dot{y}$, so

$$\begin{bmatrix} x(t) \\ \dot{x}(t) \\ y(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 & 0 & 0 \\ 0 & e^{-2t} & 0 & 0 \\ 0 & 0 & e^{-2t} & 0 \\ 0 & 0 & 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x(0) \\ \dot{x}(0) \\ y(0) \\ \dot{y}(0) \end{bmatrix}.$$

If we had the motivation, computing the first row would give an analytic solution for $x(t)$, and the third row a solution for $y(t)$, given any initial data.