Linear Systems: Qualitative Analysis

Ideas: The eigenvalues and (number of) eigenvectors of a linear system allow us to understand the behavior of the system even before writing down the exact solution. In two dimensions, with real eigenvalues we can have stable nodes (negative eigenvalues), unstable nodes (positive eigenvalues), saddles (mixed eigenvalues), or degenerate nodes (repeated eigenvalue with geometric multiplicity one). With complex eigenvalues, we can have a stable focus (negative real part), an unstable focus (positive real part), or a center (zero real part)—these may be detected by analyzing These are classified by a handful of "basic" matrices, such that every matrix is similar to one "basic" matrix.

What is the type of a matrix similar to each of the following?

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

(c) $\begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix}$

(d) $\begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$

(e) $\begin{bmatrix} \sqrt{\pi} & 0 \\ 1 & \sqrt{\pi} \end{bmatrix}$

(f) $\begin{bmatrix} 1 & x \\ 0 & -2 \end{bmatrix}$

Sketch a phase portrait for each (in some coordinate system).

(a) This has distinct, positive eigenvalues, so it is an unstable node. There is a dominant growth direction.

(b) This has positive trace, and positive upper-right entry, so it is a clockwise unstable focus.

(c) This has repeated negative eigenvalues, so it is a stable node. The off-diagonal entry tells us all orbits approach the origin at the same angle.

(d) This has a positive and a negative eigenvalue, so it is a saddle point.

(e) This isn't in a canonical form, but we can see it has a repeated positive eigenvalue, but only one (linearly independent) eigenvector, so the only type it can have is an unstable node with an off-diagonal entry, like (c).

(f) Again, this isn't in a canonical form, but we can see it has one positive and one negative eigenvalue, so whatever the value of x, it is a saddle point.

What can we conclude about the type of a system with matrix A from the following information?

(a) The trace of A is zero

(b) The determinant of A is the square of half the trace of A

(c) A has two distinct one-dimensional invariant subspaces

(d) A has no one-dimensional invariant subspaces

(a) The two eigenvalues differ only in sign: we can have either a saddle or a center (or degenerate behavior if the determinant vanishes).

(b) The matrix has a repeated eigenvalue (the trace), so we have a node.

(c) These correspond to eigenspaces, so we have two distinct eigenvectors, so can have either a node (stable or unstable) or a saddle.

(d) This guarantees we have either a focus or a center (some kind of rotation).