Linear Systems: Using Canonical Forms

Ideas: By computing eigenvalues and -vectors, we can determine the behavior of a linear system. This decomposes it into "blocks" which each exhibit some "ideal" kind of behavior, which is then distorted by a change of basis to give the actual behavior.

Describe the canonical blocks corresponding to the different eigenvalues and eigenspaces for the following matrices, and briefly describe the eigenspaces.

(a)

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 2 & 2 & 3 & 4 & 5 \\ 1 & 3 & 3 & 4 & 5 \\ 1 & 2 & 4 & 4 & 5 \\ 1 & 2 & 3 & 5 & 5 \\ 1 & 2 & 3 & 4 & 6 \end{bmatrix}$$

(a) Here, we observe that the matrix is block triangular, with upper left block $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. As the span of the first two standard basis vectors is an invariant subspace under the action of the matrix, the eigenvalues of the block will be eigenvalues of the full matrix. These are $\lambda = 1 \pm i$. An eigenvector for $1+i$ is $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

The lower right block is the same as the upper left, so $A - \lambda I$ has rank 2 for both eigenvalues of the block. Thus, we will be able to find another 2-dimensional invariant subspace with respect to which our matrix acts as $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ (for this exercise, we will not compute it).

(b) Letting this matrix be A, we see that $A = I + S$, where $S = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$. Thus, the eigenvalues of A are $1+\mu$ for μ an eigenvalue of S (as I and S commute). But we can see that $\operatorname{rank}(S) = 1$, so it has a 4-dimensional kernel

$$\ker(S) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} : \sum_{i=1}^{5} i x_i = 0 \right\}.$$

Further, as the trace is the sum of the eigenvalues, the last eigenvalue of S must be 15, and the eigenspace must be $\operatorname{ran}(S)$, as no other one-dimensional subspace is invariant. Thus, the eigenvalues of A are 1 with eigenspace $\ker(S)$, and 16 with eigenspace $\operatorname{ran}(S)$.

What can we conclude about the various subspaces of the system PJP^{-1} where

$$P = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 0 & 7 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}, \qquad J = \begin{bmatrix} -1 & 3 & 0 & 0 & 0 \\ -3 & -1 & 0 & 0 & 0 \\ 0 & 0 & -6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}?$$

The solution has $E^E = \text{span}\left\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right\}$, $E^S = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 7 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right\}$, and $E^U = \text{span}\left\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}\right\}$. There is a two-dimensional invariant subspace where the system behaves as a stable focus, and three one-dimensional invariant subspaces, one stable, one equilibrium, and one unstable.