Linear Systems: Jordan Decomposition

Ideas: The collection of (possibly generalized) eigenvectors is sufficient to determine the Jordan canonical form of a matrix (up to reordering of the blocks). To find it, we construct chains of generalized eigenvectors until we have enough for a basis.

> Compute the Jordan canonical form for the matrix
>
> $$A = \begin{bmatrix} 2 & 1 & -2 & 7 & -70 \\ 0 & 2 & 1 & -8 & 110 \\ 0 & 0 & 2 & 0 & 8 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

Here, we begin by observing that, as a triangular matrix, we clearly have a single eigenvalue 2, with algebraic multiplicity 5. As

$$A - 2I = \begin{bmatrix} 0 & 1 & -2 & 7 & -70 \\ 0 & 0 & 1 & -8 & 110 \\ 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

we know there are two (proper) eigenvectors, as this has rank 2 We can then compute

$$(A - 2I)^2 = \begin{bmatrix} 0 & 0 & 1 & -8 & 101 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

which clearly has rank 1, so there are also two "level two" generalized eigenvectors. We know there must be at least one "level three" generalized eigenvector, and we can only have one more linearly independent direction, so there is exactly one.

Now, the "level three" is in a chain with a "level two" and a proper eigenvector (that is, if w is the "level three", there are $u = (A-2I)w$ and $v = (A-2I)u$ such that $(A-2I)^3 w = (A-2I)^2 u = (A-2I)v = 0$), and the other "level two" must be part of an independent chain, so we have the Jordan form

$$J = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

Note that we have computed no vectors whatsoever.