Nonlinear Systems: Exact Solutions

Ideas: While in general, producing exact solutions to nonlinear systems is difficult and often impossible with elementary functions, sometimes there are symmetries or structure in the problem that we can use to make it solvable.

Solve the IVP

$$\begin{cases} \dot{x} = x^2 y, \\ \dot{y} = xy^2, \\ x(0) = 1, \\ y(0) = -1. \end{cases}$$

We want to find some relationship between x and y to uncouple these equations. Both derivatives involve an xy factor, so we write

$$\frac{\dot{x}}{x} = xy = \frac{\dot{y}}{y}$$

(this is simply a true equation relating x and y, given our ODE). This means that $(\log x)' = (\log y)'$; integrating gives

$$\log x = \log y + C,$$

or $x = Cy$. Looking at our initial conditions, we get $C = -1$; then our system is equivalent to the uncoupled system

$$\begin{cases} \dot{x} = -x^3, \\ \dot{y} = -y^3, \\ x(0) = 1, \\ y(0) = -1. \end{cases}$$

These are separable equations, so we can solve to get

$$\begin{cases} x(t) = \frac{1}{\sqrt{1+2t}}, \\ y(t) = \frac{-1}{\sqrt{1+2t}}. \end{cases}$$

Solve the IVP

$$\begin{cases} \dot{x} = \frac{1}{3}\left(e^x e^{2y} - 2e^{-x}e^y\right), \\ \dot{y} = \frac{1}{3}\left(e^x e^{2y} + e^{-x}e^y\right), \\ x(0) = x_0, \\ y(0) = y_0. \end{cases}$$

To uncouple the equations, we see that $u = x + 2y$ and $v = x - y$ are both relevant quantities in both derivatives. Further, if we make the change of variables, we get the equivalent system

$$\begin{cases} \dot{u} = e^u, \\ \dot{v} = -e^{-v}, \\ u(0) = x_0 + 2y_0, \\ v(0) = x_0 - y_0. \end{cases}$$

This is a separable system, with solution

$$\begin{cases} u(t) = -\log\left(e^{x_0+2y_0} - t\right), \\ v(t) = \log\left(e^{x_0-y_0} - t\right). \end{cases}$$

As $x = \frac{1}{3}(u - 2v)$ and $y = \frac{1}{3}(u - v)$, we get

$$\begin{cases} x(t) = \frac{-1}{3}\left(\log\left(e^{x_0+2y_0} - t\right) + 2\log\left(e^{x_0-y_0} - t\right)\right), \\ y(t) = \frac{-1}{3}\left(\log\left(e^{x_0+2y_0} - t\right) + \log\left(e^{x_0-y_0} - t\right)\right). \end{cases}$$

Find a decoupled system equivalent to

$$\begin{cases} x' = \cos y, \\ y' = -\cos x, \\ x(0) = 0, \\ y(0) = 0. \end{cases}$$

We can write

$$\cos(x)x' + \cos(y)y' = 0$$

and integrate to get $\sin x + \sin y = C = 0$ given the initial data. Near the origin, this implies that $x = \arcsin(\sin(-y)) = -y$. Thus, our system is equivalent to

$$\begin{cases} x' = \cos x, \\ y' = -\cos y, \\ x(0) = 0, \\ y(0) = 0. \end{cases}$$

These equations are separable, so we could integrate them at this point.