Nonlinear Systems: Parameter Dependence

Ideas: We are interested in the dependence of differential equations on the initial data and any parameters present; in particular, we want open neighborhoods in "parameter space" for which the dependence is continuous. When possible, this can be done by inspecting the solution.

> Consider the differential equation
> $$\dot{u} = u - \epsilon u^2,$$
> with initial condition u_0. Given (u_0, ϵ), what time interval does the solution exist on? Determine open subsets $E \subseteq \mathbb{R}^2$ and values a such that, if $(t, u_0, \epsilon) \in [-a, a] \times E$, the differential equation's solution depends continuously on the parameters.

We begin by computing the solution. This equation is separable, so we compute the solution with the integral

$$\int_{u_0}^{u} \frac{du}{u(1-\epsilon u)} = \int_0^t dt.$$

We use a partial fraction decomposition on the left hand side, writing

$$\frac{1}{u(1-\epsilon u)} = \frac{1}{u} + \frac{\epsilon}{1-\epsilon u},$$

then compute the integral as

$$[\log u]_{u_0}^{u} - [\log w]_{1-\epsilon u_0}^{1-\epsilon u} = \log\left(\frac{u}{1-\epsilon u}\right) - \log\left(\frac{u_0}{1-\epsilon u_0}\right).$$

The right hand side is, of course, t, so we have

$$\frac{u}{1-\epsilon u} = \frac{u_0 e^t}{1-\epsilon u_0}.$$

Solving for u yields

$$u(t) = \frac{u_0 e^t}{1 + \epsilon u_0 (e^t - 1)}.$$

Now, inspecting this solution, it blows up if we ever have

$$1 + \epsilon u_0 (e^t - 1) = 0,$$

which we can rephrase as

$$e^t = 1 - \frac{1}{\epsilon u_0}$$

(assuming that is nonzero; the solution clearly exists for all time if it is zero). If $0 < \epsilon u_0 < 1$, the right hand side is negative, so the condition will never be reached for any t. If $\epsilon u_0 < 0$ or $\epsilon u_0 > 1$, the right hand side is positive, so the condition will be reached, and the solution will exist for a semi-infinite period of time. In the first case, the interval of existence will be $\left(-\infty, \log\left(1 - \frac{1}{\epsilon u_0}\right)\right)$; in the second, it will be $\left(\log\left(1 - \frac{1}{\epsilon u_0}\right), \infty\right)$. If $\epsilon u_0 = 1$, we see our solution is constant, hence exists for all time.

Based on the previous concerns, so long as we take the set E to be bounded, we will be able to find an a giving a region of continuity. Such an a will be anything strictly less than the minumum of the infimum of $\log\left(1 - \frac{1}{\epsilon u_0}\right)$ over the intersection of E and $\{\epsilon u_0 < 0\}$ and the infimum of $-\log\left(1 - \frac{1}{\epsilon u_0}\right)$ over the intersection of E and $\{\epsilon u_0 > 1\}$. This is because our solution formula depends continuously on all parameters and t wherever it is finite.