First Order Equations

1 Describe the solution of the equation

$$u_t + cu_x + u = x^2$$

for c a constant and initial condition $u(0,x) = u_0(x)$.

We first rewrite the equation as

$$(1,c) \cdot \nabla u = x^2 - u.$$

A change of variables to $\xi = x + ct$ and $\eta = x - ct$ then allows us to write $x = \frac{1}{2}(\xi + \eta)$ and $t = \frac{1}{2c}(\xi - \eta)$. Note that this is not an orthogonal transformation, but one which allows easy representation of x and t. Then

$$u_\xi = u_x x_\xi + u_t t_\xi = \frac{1}{2}u_x + \frac{1}{2c}u_t = \frac{1}{2c}(u_t + cu_x,)$$

so our differential equation may be written

$$u_\xi = \frac{c}{2}(\xi + \eta)^2 - 2cu.$$

The ODE $u_\xi = -2cu$ has as a general solution $f(\eta)\exp(-2c\xi)$. The ODE $u_\xi + 2cu = \frac{c}{2}(\xi + \eta)^2$ has a particular solution of the form $A(\eta)\xi^2 + B(\eta)\xi + C(\eta)$, where $2cA(\eta) = \frac{c}{2}$, $2A(\eta) + 2cB(\eta) = c\eta$, and $B(\eta) + 2cC(\eta) = \frac{c}{2}\eta^2$. Solving these iteratively, we get $A(\eta) = \frac{1}{4}$, $B(\eta) = \frac{1}{2}\eta - \frac{1}{4c}$, and $C(\eta) = \frac{1}{4}\eta^2 - \frac{1}{4c}\eta + \frac{1}{8c^2}$. Thus, the general solution of the PDE (in ξ-η coordinates) is

$$u(\xi,\eta) = f(\eta)e^{-2c\xi} + \frac{1}{4}\xi^2 + \frac{1}{2}\xi\eta + \frac{1}{4}\eta^2 - \frac{1}{4c}\xi - \frac{1}{4c}\eta + \frac{1}{8c^2}.$$

Converting back to x-t coordinates, we have

$$u(t,x) = f(x-ct)e^{-2cx}e^{-2c^2t} + x^2 - \frac{1}{2c}x + \frac{1}{8c^2}.$$

Then $u(0,x) = f(x)e^{-2cx} + x^2 - \frac{1}{2c}x + \frac{1}{8c^2} = u_0(x)$, so

$$f(s) = e^{2cs}\left(u_0(s) - s^2 + \frac{1}{2c}s - \frac{1}{8c^2}\right).$$

This gives the solution to the initial value problem

$$u(t,x) = e^{-4c^2t}\left(u_0(x-ct) - (x-ct)^2 + \frac{1}{2c}(x-ct) - \frac{1}{8c^2}\right) + x^2 - \frac{1}{2c}x + \frac{1}{8c^2}.$$

2 Describe the solution to the equation

$$u_t + 2uu_x = 0$$

with initial condition

$$u(0,x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases}$$

Discuss the characteristic curves.

We first rewrite the equation as

$$(1, 2u) \cdot \nabla u = 0.$$

Thus, the characteristic curves are orthogonal to $(1, 2u)$, hence satisfy

$$x'(t) = 2u(t, x(t)) = 2u(0, x(0)),$$

and hence

$$x(t) = 2u(0, x(0))t + x(0).$$

If $x(0) > 0$, then we have a line $x(t) = 2t + x(0)$.

The "characteristic curves" for $x(0) < 0$ are constant. So, if (x, t) with $t > 0$ does not lie along one of the lines above, it must come from a characteristic curve emanating from the origin (a rarefaction event). For such a point, $x = 2u(t, x)t$, so $u(t, x) = \frac{x}{2t}$. Thus, our solution is

$$u(t,x) = \begin{cases} 1, & x > 2t, \\ \frac{x}{2t}, & x < 2t. \end{cases}$$

However, we can see that this is discontinuous as we approach the $x < 0$ semiaxis!