ODE Review

First Order Equations Find solutions (if possible) to the following equations for $t > 0$ given the initial condition $u(0) = u_0$.

(a) $2u' = tu$

(b) $uu' = (3-t)^{-1}$

(c) $u' + 4u = -u^2$

(d) $(t-2u)u' + (u-2t) = 0$

(a) As a linear first order ODE, this is a standard integrating factor problem; we want a function μ such that $\mu' = \frac{t}{2}\mu$, meaning $\mu = \exp(t^2/4)$. Then our DE is $\left[e^{t^2/4}u\right]' = 0$, so $u = Ce^{-t^2/4}$ for some constant C. Clearly, the solution we seek is then $u = u_0 e^{-t^2/4}$.

(b) Here, we see we have a differential equation which is linear in $v = u^2$: since $v' = 2uu'$, we have $v' = \frac{1}{2(3-t)}$, so $v = C - \frac{1}{2}\ln(3-t)$. The initial condition here is $v(0) = u_0^2$, so we get $C = u_0^2 + \frac{1}{2}\ln 3$, for a final answer of $u = \operatorname{sign}(u_0)\sqrt{u_0^2 + \frac{1}{2}\ln\left(\frac{3}{3-t}\right)}$.

(c) We can be tricky here, writing the equation as

$$\frac{-u' - 4u}{u^2} = 1,$$

and observing that with an integrating factor of $\mu = e^{-4t}$ we have $\left[\frac{e^{-4t}}{u}\right]' = e^{-4t}$, or $e^{-4t} = \left(C - \frac{1}{4}e^{-4t}\right)u$. Then $1 = (C - \frac{1}{4})u_0$, so $C = \frac{1}{u_0} + \frac{1}{4}$, and we have a final solution of $u = \frac{4u_0 e^{-4t}}{4 + u_0(1-e^{-4t})}$.

(d) We recall the multivariable chain rule: if $F(t,u)$ is some function, then $\frac{d}{dt}F = \frac{\partial}{\partial t}F + \frac{\partial}{\partial u}F\frac{du}{dt}$. So, if the coefficient of u' and the other term are compatible (in the sense their mixed partials agree), we can recover the function F (this is the motivation behind exact equations), which will be constant along a solution curve. Here, $\frac{\partial}{\partial t}[t-2u] = 1 = \frac{\partial}{\partial u}[u-2t]$, so we can find such an F.

We know $\frac{\partial}{\partial u}F = t - 2u$, so integrating gives $F = tu - u^2 + g(t)$ for some function $g(t)$. Differentiating gives $\frac{\partial}{\partial t}F = u + g'(t)$, so $g'(t) = -2t$, and we can choose $g(t) = -t^2$ (setting the integration constant to zero for convenience).

Thus, $F(t,u) = tu - u^2 - t^2$. Our solution is a curve along which this is constant, with value $F(0,u_0) = -u_0^2$. That is, it is given implicitly by the equation $tu - u^2 - t^2 + u_0^2 = 0$. A little work shows this is the equation of an ellipse, and our solution will be an arc of that ellipse.