Reflections

1 Find all solutions to the following mixed Dirichlet-Neumann problem for $t \in \mathbb{R}$ and $x \in [0,1]$:

$$\begin{cases} u_{tt} - 16u_{xx} = 0 \\ u(0,x) = 16x^2(1-x)^2 \\ u_t(0,x) = 0 \\ u_x(t,0) = 0 \\ u(t,1) = 0. \end{cases}$$

First, we find a solution by extending our initial data to the whole real line. The Neumann condition at $x = 0$ means our data must be even, while the Dirichlet condition at $x = 1$ means it must be odd about $x = 1$. The zero function satisfies both these properties already. We must have $\phi(-x) = \phi(x)$ and $\phi(1-x) = -\phi(1+x)$. These imply that

$$\phi(x+4) = -\phi(-x-2) = -\phi(x+2) = \phi(x),$$

so ϕ must be extended to a 4-periodic function. We may write this as

$$\phi(x) = \begin{cases} -16(x+2)^2(-x-1)^2, & x \in [-2,-1] \\ 16(x+1)^2(-x)^2, & x \in [-1,0], \\ 16x^2(1-x)^2, & x \in [0,1], \\ -16(x-1)^2(2-x)^2, & x \in [1,2], \\ \text{extended to be 4-periodic.} \end{cases}$$

In fact, we may simplify our notation by using the floor and integer part functions, writing

$$\phi(x) = 16[x]^2(1-[x])^2(-1)^{\lfloor \frac{x+1}{2} \rfloor}$$

(the fundamental region up to sign is $[0,1]$, and the sign changes every two integers, and is positive on $[-1,1]$). The D'Alembert's solution is then

$$u(t,x) = \frac{1}{2}\left(\phi(x-4t) + \phi(x+4t)\right).$$

Now, we consider the question of uniqueness. Suppose w solves the same problem except with zero initial condition as well, and define the energy function $E(t) = \frac{1}{2}\int_0^1 w_t^2 + 16w_x^2 \, dx$. We then have

$$E'(t) = \int_0^1 w_t w_{tt} + 16 w_x w_{xt} \, dx.$$

Integrating by parts, we have

$$E'(t) = [16 w_x w_t]_{x=0}^{x=1} + \int_0^1 w_t(w_{tt} - 16w_{xx}) \, dx.$$

Now, the Neumann condition at zero means the left boundary term vanishes, the Dirichlet condition at one means $w_t(t,1) = 0$ so the right boundary term vanishes, and the PDE itself means the integral term vanishes. Thus, $E'(t) = 0$, so the energy is constant. Since $E(0) = 0$ by the initial data, by the usual argument $w \equiv 0$. It follows that the D'Alembert's solution above is the unique solution to the problem.