Wave Equations

1 Solve the wave equation

$$u_{tt} - 9u_{xx} = 0$$

subject to the initial conditions $u(0,x) = \sin(3x)$ and $u_t(0,t) = 1$.

We set the characteristic variables $\xi = x + 3t$ and $\eta = x - 3t$, so that $u_x = u_\xi + u_\eta$ and $u_t = 3u_\xi - 3u_\eta$, and therefore $u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$ and $u_t = 9u_{\xi\xi} - 18u_{\xi\eta} + 9u_{\eta\eta}$. Thus, our equation becomes $-36u_{\xi\eta} = 0$, with a solution of $u(\xi,\eta) = f(\xi) + g(\eta)$.

Changing back to our original variables, this is $u(t,x) = f(x+3t) - g(x-3t)$. We have initial conditions $\sin(3x) = f(x) + g(x)$ and $1 = 3f'(x) - 3g'(x)$. Integrating the latter, we get $\frac{x}{3} + f(0) - g(0) = f(x) - g(x)$. Solving, we have $f(x) = \frac{1}{2}\left[\sin(3x) + \frac{x}{3}\right] + \frac{1}{2}[f(0) - g(0)]$ and $g(x) = \frac{1}{2}\left[\sin(3x) - \frac{x}{3}\right] - \frac{1}{2}[f(0) - g(0)]$. This tells us that, in general, our solution is

$$u(t,x) = \frac{1}{2}[\sin(3x+9t) + \sin(3x-9t)] + t.$$

If we want a slightly cleaner solution, we may use the identity $2\sin a \cos b = \sin(a+b) + \sin(a-b)$ to write this as

$$u(t,x) = \sin(3x)\cos(9t) + t.$$

This shows an interesting feature, related to the "standing wave" phenomenon: because the initial position was sinusoidal, its contribution to the wave shape oscillates in place (individual "particles" still travel along the characteristics). Of course, the initial velocity was not, so there is an overall growth term as well.

2 Prove the problem

$$\begin{cases} u_{tt} + 2\zeta u_t = c^2 u_{xx} + F(t,x), \\ u(0,x) = u_0(x), \\ u_t(0,x) = v_0(x). \end{cases}$$

has a unique solution (if one exists) provided the spatial derivatives of the solution vanish as x tends to $\pm\infty$. Here $\zeta \geq 0$ is a fixed constant, and F a given function.

If u and $\bar{u}$ are two solutions to the above problem, then their difference $w = u - \bar{u}$ solves the corresponding homogeneous problem

$$\begin{cases} w_{tt} + 2\zeta w_t - c^2 w_{xx} = 0, \\ w(0,x) = 0, \\ w_t(0,x) = 0. \end{cases}$$

Define the energy $E(t) = \frac{1}{2}\int_{\mathbb{R}} w_t^2 + c^2 w_x^2 \, dx$. Then we have

$$E'(t) = \int_{\mathbb{R}} w_t u_{tt} + c^2 w_x w_{xt} \, dx = \int_{\mathbb{R}} w_t \left(w_{tt} - c^2 w_{xx}\right) dx$$

by integration by parts (and observing the boundary terms vanish, by the condition on w_x at infinity). Thus, we have

$$E'(t) = -2\zeta \int_{\mathbb{R}} w_t^2 \leq 0.$$

Now, $E(t) \geq 0$ and $E'(t) \leq 0$. We may directly compute $E(0) = \frac{1}{2}\int_{\mathbb{R}} w_t(0,x)^2 + w_x(0,x)^2 \, dx = 0$. Thus, we must have $E(t) = 0$ for all $t \geq 0$ ($w(0,x) = 0$, so its x derivative also vanishes). This tells us that w_t and w_x are both identically zero (otherwise they would contribute positively to the integral somewhere). So, $\nabla w \equiv 0$, so $w(t,x) \equiv 0$, as it is constant and zero for $t = 0$.

3 Find the general solution to

$$u_{tt} + (1-t)u_{xt} - tu_{xx} = 0.$$

Hint: factor the differential operator.

We factor the differential operator as $(\partial_t - t\partial_x)(\partial_t + \partial_x)$. Thus, making the definition $v = u_t + u_x$, we see we have the chain of first order equations

$$\begin{cases} u_t + u_x = v, \\ v_t - tv_x = 0. \end{cases}$$

Then $v(t,x) = f(x+\frac{1}{2}t^2)$ for some function f. So, we have the equation $u_t + u_x = f(x+\frac{1}{2}t^2)$. Making the substitutions $\xi = x - t$ and $\eta = x + t$, we have $u_\xi = f(\frac{1}{8}\xi^2 - \frac{1}{4}\xi\eta + \frac{1}{8}\eta^2 + \frac{1}{2}\xi + \frac{1}{2}\eta)$. Then we have the general solution

$$u(t,x) = g(x-t) + \int_0^{x-t} f\left(\frac{1}{8}\xi^2 - \frac{1}{4}\xi(x+t-2) + \frac{1}{8}x^2 + \frac{1}{4}xt + \frac{1}{8}t^2 + \frac{1}{2}x + \frac{1}{2}t\right) d\xi.$$