Generalized Eigenvectors

Generalized Eigenvectors: Suppose A is a linear operator and $\vec{x}$ is an eigenvector for A (so $A\vec{x} = \lambda\vec{x}$). A *generalized eigenvector* of rank $m > 1$ for $\vec{x}$ is a vector $\vec{\eta}$ such that $(A - \lambda I)^m \vec{\eta} = \vec{0}$ and $(A - \lambda I)^{m-1}\vec{\eta} = \vec{x}$. That is, applying $(A - \lambda I)$ enough times results in $\vec{x}$, and so one more time gives $\vec{0}$. A linear operator can only have a generalized eigenvector of rank m if it has one each of ranks $m-1$, $m-2$, ..., and 1 (so we start at rank 1 and work our way up until we have enough). Note that while a (nonzero) scalar multiple of an eigenvector is also an eigenvector, this is *not* the case with generalized eigenvectors.

1: Find all the eigenvalues, eigenvectors, and generalized eigenvectors for the following matrices:

(a) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 11 & -1 \\ 1 & 9 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

(e) $\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & -2 & 0 \\ 2 & -3 & 0 \\ -1 & 1 & -1 \end{bmatrix}$

(a) The characteristic equation here is $\lambda^2 = 0$, and $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector. A generalized eigenvector is $\vec{\eta} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, as it solves the equation $A\vec{\eta} = \vec{x}$ (this is essentially what we are doing all the time, although most other cases require some computation, as below).

(b) Here, the characteristic equation is $(1-\lambda)^2 = 0$, so we have 1 as our only eigenvalue. An eigenvector for 1 is $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. So, we would like to solve the equation $(A - I)\vec{\eta} = \vec{x}$ (then $(A-I)^2\vec{\eta} = \vec{0}$ automatically). That is, $\begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. This has one solution $\vec{\eta} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ (in general, any vector $\begin{bmatrix} a \\ 2 \end{bmatrix}$ will work). So, $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ (this factorization is called the Jordan decomposition and is a generalization of diagonalization).

(c) Here, the characteristic polynomial is $(11-\lambda)(9-\lambda) + 1 = 100 - 20\lambda + \lambda^2 = (10-\lambda)^2 = 0$. An eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So, we want to solve the equation $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\vec{\eta} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. One solution is $\vec{\eta} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. (If you're curious, the Jordan decomposition is $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 10 & 1 \\ 0 & 10 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ in this case).

(d) Here, our characteristic polynomial is $(1-\lambda)^2(2-\lambda) = 0$. The eigenvectors for $\lambda = 2$ are in the kernel of $\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, so are in the span of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. The eigenvectors for $\lambda = 1$ are in the kernel of $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and so are in the span of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. We need a generalized eigenvector for 1, which is a solution to $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\vec{\eta} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, so $\vec{\eta} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

(e) Here, our characteristic polynomial is $-\lambda^3 = 0$, so our only eigenvalue is $\lambda = 0$. The only eigenvectors are in the span of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Our rank one generalized eigenvector is then $\begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}$, and our rank two generalized eigenvector is

$\begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \end{bmatrix}$ (note that A applied to this rank two generalized eigenvector yields the rank one generalized eigenvector, so A^2 yields the eigenvector and A^3 yields the zero vector).

(f) Here, our characteristic polynomial is

$$\begin{vmatrix} 1-\lambda & -2 & 0 \\ 2 & -3-\lambda & 0 \\ -1 & 1 & -1-\lambda \end{vmatrix} = (-1-\lambda)\begin{vmatrix} 1-\lambda & -2 \\ 2 & -3-\lambda \end{vmatrix} = (-1-\lambda)\,((1-\lambda)(-3-\lambda)+4) = (-1-\lambda)(1+2\lambda+\lambda^2) = (-1-\lambda)^3 = 0,$$

so our only eigenvalue is $\lambda = -1$. Our eigenvectors are in the kernel of $\begin{bmatrix} 2 & -2 & 0 \\ 2 & -2 & 0 \\ -1 & 1 & 0 \end{bmatrix}$, so are in the span of $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. The eigenvector $\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ (which is a linear combination of the two basis eigenvectors we found) has a generalized eigenvector of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ (note that neither basis eigenvector has a generalized eigenvector; the matrix $A+I$ has rank one, so only one of the eigenvectors has a generalized eigenvector associated to it).