Math 4B
Worksheet 10a

Generalized Eigenvectors

The Punch Line: Generalized eigenvectors allow us to write down the solution to differential equations where there are not enough eigenvectors to form solutions like we have before.

Setup: If λ is an eigenvalue of A where the eigenspace has one fewer dimension than the multiplicity of λ in the characteristic equation of A, then given a vector u such that $(A - \lambda I)u = v$ for v an eigenvector of A, then $e^{\lambda t}(tv + u)$ is a solution of $x' = Ax$.

We can write the solution to a DE as a fundmental matrix $\Psi(t)\vec{c}$, or more specifically $\Phi(t)\vec{x}(0)$. We can recover $\Phi(t) = \Psi(t)\Psi(0)^{-1}$ (although often computing $\Psi(0)^{-1}$ is more computationally difficult than solving the equation $\Psi(0)\vec{c} = \vec{x}(0)$ for $\vec{c}$ by row reduction).

1: Solve the following DEs (if initial conditions are given, use them, otherwise give the general solution). Write a fundamental matrix $\Psi(t)$ such that $x(t) = \Psi(t)\vec{c}$ (it might be a good idea to find a $\Phi(t)$ such that $x(t) = \Phi(t)x(0)$—such as $\Psi(t)\Psi(0)^{-1}$).

(a) $x' = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix} x$

(b) $x' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} x$

(c) $x' = \begin{bmatrix} -9 & 4 \\ -16 & 7 \end{bmatrix} x$

(d) $x' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} x$

(e) $x' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} x$ (the eigenvalues are $\pm i$, each with multiplicity two).