Generalized Eigenvectors

The Punch Line: Generalized eigenvectors allow us to write down the solution to differential equations where there are not enough eigenvectors to form solutions like we have before.

Setup: If λ is an eigenvalue of A where the eigenspace has one fewer dimension than the multiplicity of λ in the characteristic equation of A, then given a vector u such that $(A-\lambda I)u = v$ for v an eigenvector of A, then $e^{\lambda t}(tv+u)$ is a solution of $x' = Ax$.

We can write the solution to a DE as a fundmental matrix $\Psi(t)\vec{c}$, or more specifically $\Phi(t)\vec{x}(0)$. We can recover $\Phi(t) = \Psi(t)\Psi(0)^{-1}$ (although often computing $\Psi(0)^{-1}$ is more computationally difficult than solving the equation $\Psi(0)\vec{c} = \vec{x}(0)$ for $\vec{c}$ by row reduction).

1: Solve the following DEs (if initial conditions are given, use them, otherwise give the general solution). Write a fundamental matrix $\Psi(t)$ such that $x(t) = \Psi(t)\vec{c}$ (it might be a good idea to find a $\Phi(t)$ such that $x(t) = \Phi(t)x(0)$—such as $\Psi(t)\Psi(0)^{-1}$).

(a) $x' = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix} x$

(b) $x' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} x$

(c) $x' = \begin{bmatrix} -9 & 4 \\ -16 & 7 \end{bmatrix} x$

(d) $x' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} x$

(e) $x' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} x$ (the eigenvalues are $\pm i$, each with multiplicity two).

(a) We compute the characteristic equation as $(5-\lambda)^2 - 1 = \lambda^2 - 10\lambda + 24 = (\lambda-4)(\lambda-6) = 0$. We compute $v_4 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v_6 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ as the eigenvectors for 4 and 6 respectively. So, the general solution is

$$x(t) = C_1 e^{4t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{6t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} e^{4t} & e^{6t} \\ e^{4t} & -e^{6t} \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}.$$

We then have $\Psi(0)^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Then $\Phi(t) = \Psi(t)\Psi(0)^{-1} = \frac{1}{2}\begin{bmatrix} e^{4t}+e^{6t} & e^{4t}-e^{6t} \\ e^{4t}-e^{6t} & e^{4t}+e^{6t} \end{bmatrix}$.

(b) We compute the characteristic equation as $(1-\lambda)(3-\lambda)+1 = \lambda^2 - 4\lambda + \lambda^2 = 0$, so $\lambda = 2$ (with multiplicity 2). We find $v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ as an eigenvector for 2, but no second eigenvector. Then we look at $(A-2I)u = v$, finding that we must have $u_1 + u_2 = -1$, so $u = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ is a generalized eigenvector. We can then write the general solution

$$x(t) = C_1 e^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^{2t}\left(t\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} e^{2t} & te^{2t} \\ -e^{2t} & -(t+1)e^{2t} \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}.$$

We then have $\Psi(0) = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$, so $\Psi(0)^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$, so $\Phi(t) = \begin{bmatrix} e^{2t} & te^{2t} \\ -e^{2t} & (1+t)e^{2t} \end{bmatrix} = e^{2t}\begin{bmatrix} 1-t & -t \\ t & 1+t \end{bmatrix}$.

(c) We find the characteristic polynomial $(-9-\lambda)(7-\lambda)+64 = \lambda^2+2\lambda+1 = (\lambda+1)^2 = 0$. We then get $v = \begin{bmatrix}1\\2\end{bmatrix}$ as a eigenvector for -1. Then solving $(A+I)u = v$ we get $-2u_1+u_2 = \frac{1}{4}$, so $u = \begin{bmatrix}0\\\frac{1}{4}\end{bmatrix}$ is a generalized eigenvector for eigenvalue -1. Then we can write the general solution as

$$x(t) = C_1e^{-t}\begin{bmatrix}1\\2\end{bmatrix} + C_2e^{-t}\left(t\begin{bmatrix}1\\2\end{bmatrix}+\begin{bmatrix}0\\\frac{1}{4}\end{bmatrix}\right).$$

Then $\Psi(t) = e^{-t}\begin{bmatrix}1 & t\\2 & 2t+\frac{1}{4}\end{bmatrix}$, so $\Psi(0)^{-1} = \begin{bmatrix}1 & 0\\-8 & 4\end{bmatrix}$. So, $\Phi(t) = e^{-t}\begin{bmatrix}1-8t & 4t\\-16t & 1+8t\end{bmatrix}$.

(d) We see that 1 is the sole eigenvalue of the matrix, and $v_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ and $v_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$ are eigenvectors. Examining $(A-I)u = v$, we see that v_1 does not have generalized eigenvectors, but for v_2 we have $u = \begin{bmatrix}0\\0\\\frac{1}{2}\end{bmatrix}$. So, our general solution is

$$x(t) = C_1e^t\begin{bmatrix}1\\0\\0\end{bmatrix} + C_2e^t\begin{bmatrix}0\\1\\0\end{bmatrix} + C_3e^t\left(t\begin{bmatrix}0\\1\\0\end{bmatrix}+\begin{bmatrix}0\\0\\\frac{1}{2}\end{bmatrix}\right) = e^t\begin{bmatrix}1 & 0 & 0\\0 & 1 & t\\0 & 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}C_1\\C_2\\C_3\end{bmatrix}.$$

Then $\Psi(0)^{-1} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 2\end{bmatrix}$, so $\Phi(t) = e^t\begin{bmatrix}1 & 0 & 0\\0 & 1 & 2t\\0 & 0 & 1\end{bmatrix}$.

(e) We compute the eigenvectors $\begin{bmatrix}1\\i\\0\\0\end{bmatrix}$ and $\begin{bmatrix}0\\0\\i\\1\end{bmatrix}$ for eigenvalue i. Then we can compute a general solution as

$$x(t) = C_1\begin{bmatrix}\cos t\\-\sin t\\0\\0\end{bmatrix} + C_2\begin{bmatrix}\sin t\\\cos t\\0\\0\end{bmatrix} + C_3\begin{bmatrix}0\\0\\-\sin t\\\cos t\end{bmatrix} + C_4\begin{bmatrix}0\\0\\\cos t\\\sin t\end{bmatrix} = \begin{bmatrix}\cos t & \sin t & 0 & 0\\-\sin t & \cos t & 0 & 0\\0 & 0 & -\sin t & \cos t\\0 & 0 & \cos t & \sin t\end{bmatrix}\begin{bmatrix}C_1\\C_2\\C_3\\C_4\end{bmatrix}.$$

Then $\Psi(0)^{-1} = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 1 & 0\end{bmatrix}$, so $\Phi(t) = \begin{bmatrix}\cos t & \sin t & 0 & 0\\-\sin t & \cos t & 0 & 0\\0 & 0 & \cos t & -\sin t\\0 & 0 & \sin t & \cos t\end{bmatrix}$.