Phase Portraits

The Punch Line: We can tell the behavior of a two dimensional system by examining the eigenvalues.

Setup: If A is a 2×2 matrix, then if it has two positive eigenvalues it is a source (unstable), if it has two negative eigenvalues it is a sink (stable), and if it has one of each it is a saddle (unstable). If both eigenvalues are nonzero but have different values we have a nodal source or sink, if they are equal and there are two eigenvectors we have a star node, if they are equal and there is only one eigenvector we have a degenerate node. If we have complex eigenvalues, we have a stable or unstable spiral (depending on if the sign of the real part is negative or positive, respectively), or a center if they are both purely imaginary. Finally, if one eigenvalue is zero we have infinitely many equilibrium points, and are stable or unstable according to the nonzero eigenvalue (if both eigenvalues are nonzero, we have only constant solutions).

1: Classify the following DEs according to their phase plane behavior. Are they stable, unstable, or neither?

(a) $x' = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix} x$

(b) $x' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} x$

(c) $x' = \begin{bmatrix} -9 & 4 \\ -16 & 7 \end{bmatrix} x$

(d) $x' = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix} x$

(e) $x' = \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} x$

(f) $x' = \begin{bmatrix} -4 & 2 \\ -1 & -2 \end{bmatrix} x$

(a) We compute the characteristic equation as $(5-\lambda)^2 - 1 = (\lambda-4)(\lambda-6) = 0$, finding two distinct positive eigenvalues. Thus, our system is a nodal source (unstable).

(b) We find eigenvalues of $\lambda = 2$ (with multiplicity 2), and only one eigenvector (see worksheet 10a). Thus, we have a degenerate nodal source (unstable).

(c) This has eigenvalues of $\lambda = -1$ (with multiplicity 2), and only one eigenvector, so we have a degenerate nodal sink (stable).

(d) This has characteristic polynomial $\lambda^2 + 3$, so eigenvalues $\lambda = \pm i\sqrt{3}$. These are purely imaginary, so we have a center (neither stable nor unstable).

(e) This has eigenvalues $\lambda = \pm 1$ (it is triangular, so we can read them off the diagonal). So, it is a saddle point, which is unstable.

(f) This has characteristic polynomial $(-4-\lambda)(-2-\lambda) + 2 = 10 + 6\lambda + \lambda^2 = 0$, so has eigenvalues $-3 \pm i$. So, it is a spiral sink (stable).