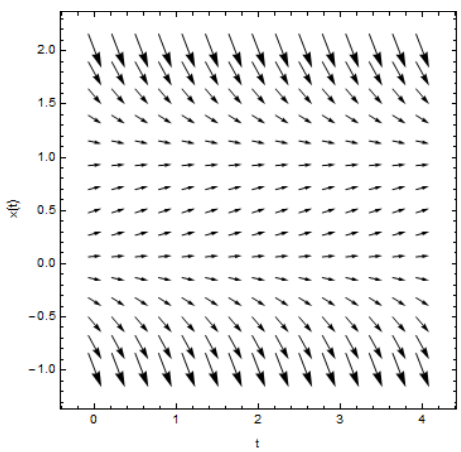
Slope Fields

The Punch Line: Slope fields allow us to visualize the behavior of a differential equation (without solving it).

1: Suppose we have the differential equation

$$x' = x(1-x),$$

involving some function $x(t)$. It has the *slope field* below:



Does this equation have solutions which

(a) constantly grow in magnitude (absolute value)?

(b) constantly shrink in magnitude?

(c) constantly grow larger (more positive)?

(d) constantly grow smaller (more negative)?

(e) are bounded (magnitude is smaller than some value for all time)?

(f) are unbounded (magnitude grows arbitrarily large)?

(g) are constant? (these are *equilibrium solutions*)

(h) grow arbitrarily large in the positive direction?

(i) grow arbitrarily large in the negative direction?

Try to describe or sketch any solutions which have these behaviors. Where in the plot do they occur?

Don't worry if you can't wholly justify your answers, or if you aren't able to tell the behavior from the plot definitively—we don't have the analytical tools to properly answer these questions yet, this is just a chance to think about what kind of information we can get from a slope field!

(a) Yes! Any solution in the bottom part of the plot (below $x = 0$) grows constantly more negative forever, so gets further from zero for all time. Also, for $0 < x < 1$, the solutions are growing constantly larger (arrows are pointed upwards), although this slows down as you get to the edges of the region. It's a nontrivial fact that these solutions don't "escape" to become larger than 1; we can't fully tell this from the plot, but it appears to be true (and turns out to be the case).

(b) Yes! Any solution in the upper part of the plot (above $x = 1$) constantly moves downwards towards $x = 1$ (although it'll turn out it'll never actually reach it). Even though they slow down as they approach $x = 1$, they will always be decreasing—again, it's nontrivial that they don't "escape".

(c) Yes! The solutions for $0 < x < 1$ are always increasing. Note that if x is between zero and one, the derivative is always positive: we can tell this from the equation, so we can be sure it's true.

(d) Yes! The solutions for $x < 0$ or $x > 1$ are always decreasing; again, we can check this in the equation.

(e) Yes! The solutions for $0 < x < 1$ all have magnitude less than 1 for all time. As it turns out, the solutions starting with $x > 1$ are also bounded in the sense that as time increases, they are always smaller than their starting values (this is assuming that we're only considering the solutions *forwards* in time, which isn't always the case!).

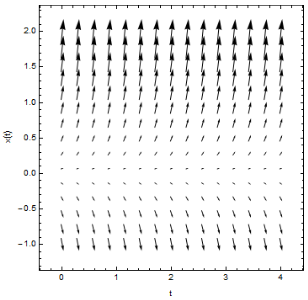
(f) Yes! The solutions for $x < 0$ get arbitrarily negative, as they are "accelerating downwards" in a sense (although this might not be an equation describing physical motion, so that terminology might not be entirely correct). Note that even though the solutions for $0 < x < 1$ are always increasing, the rate of increase slows sufficiently that they are bounded, not unbounded—again, this is something we can't tell for sure from the plot, although it suggests it.

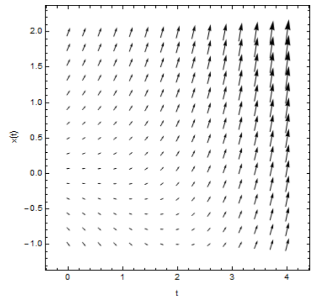
(g) Yes! The solutions for $x = 0$ and $x = 1$ are constant (this is a little hard to tell on the plot, but you can check with the equation that the derivative for these values is zero, so the solution stays there).

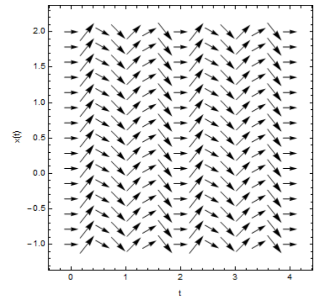
(h) No! The only increasing solutions are for $0 < x < 1$, and these are all bounded.

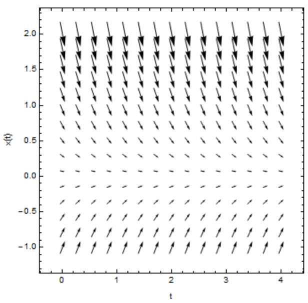
(i) Yes! The solutions for $x < 0$ become arbitrarily negative.

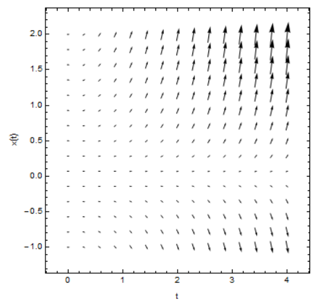
2: Match the slope fields below to the given differential equations:

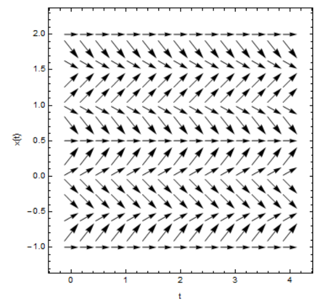

(a)


(c)


(e)


(b)


(d)


(f)

(i) $x' = x + \frac{t^2}{4}$

(ii) $x' = 4x$

(iii) $x' = tx$

(iv) $x' = -2x$

(v) $x' = \sin(2\pi x)$

(vi) $x' = \sin(2\pi t)$

(a) This goes with equation (ii), as it grows positive when x is positive, grows negative when x is negative, and does not depend on t.

(b) This goes with equation (iv), as it shrinks when x is positive and grows when x is negative (always the opposite sign), and does not depend on t.

(c) This goes with equation (i), as it depends on t, and looks somewhat like (a) when t is small, while it seems to almost uniformly increase when t is large.

(d) This goes with equation (iii), as it looks like (a) except more pronounced when t is large and more subdued when t is small.

(e) This goes with equation (vi), as it oscillates and depends on t, not x.

(f) This goes with equation (v), as it oscillates and depends on x, not t.