Math 4B
Worksheet 2

First Order Equations

The Punch Line: When we have only a single derivative, we can often make progress by recognizing the results of various derivative rules.

The Product Rule and Integrating Factors: Recall that if we have two functions f and g of the same variable x, then the derivative $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$. So, if we recognize the form on the right, we can put it back into the form on the left.

The differential equation $u'(t) = g(t)$ is "easy" to solve: integrating both sides with respect to t (starting at some t_0) gives $u(t) = u(t_0) + \int_{t_0}^{t} g(s)\,ds$ (using s as a "dummy variable" for integration). What we want is to create this situation in an equation we are given. If the equation has (or can be put into) the form

$$y' + p(t)y = g(t),$$

then the left hand side is almost the result of a product rule: it is the result of $[\mu y]'$, where $\mu(t) = e^{\int p(s)\,ds}$ (the "integral" represents any antiderivative), divided by $\mu(t)$ (which we can check is never zero). So if we multiply both sides by $\mu(t)$, we can undo the product rule and integrate, solving the equation!

1 (Integrating Factor): Can an integrating factor be used to create a product rule and solve these differential equations? If so, what is the integrating factor and solution (it's okay to leave it in integral form); if not, what is the obstruction?

(a) $y' + y = \sin(\pi t)$

(b) $xy' + y = \frac{1}{y}$

(c) $xy' + y = e^{-2x}$

(d) $u^2 + t^2 = u'$

(e) $\sqrt{u' + \sin(t)u} = t$

(f) $2uu' + u^2 = \frac{1}{3}t^3 - 1$

(g) $y'' + y = t\cos t$

(h) $\frac{y' - e^t y}{t - e^{-t}} = 1$

(i) $\cos(t)y' + y = 0$

Work Space

The Chain Rule and Separation of Variables: Recall that if we have two functions f and g, then the derivative $[f(g(x))]' = f'(g(x))g'(x)$. We're again going to look for the form on the right, to work backwards to the form on the left.

If an equation has the form

$$M(x) + N(y)y' = 0,$$

we pretty much have the result of a chain rule: the $N(y)y'$ term is the chain-rule derivative of an antiderivative of N. So, we can re-write the equation as $N(y)y' = -M(x)$, integrate both sides, and use a substitution on the left integral (namely, $u = y$, $du = y'dx$) to get an explicit answer.

2 (Separation of Variables): Are the following equations separable? If so, what are M and N, and what is the solution? If not, what is the obstruction?

(a) $y' + y = \sin(\pi t)$

(b) $yy' = x$

(c) $xy' = y(1-x)$

(d) $\cos(y)y' = 3t^2$

(e) $y' + x = y + 1$

(f) $\frac{u'}{u-1} = 1$

Challenge Problems: Of course, both of these methods are just recognizing the result of certain derivative rules. In general, if we're able to manipulate a differential equation in such a way that we recognize the result of a derivative rule, we can "undo" that rule and (hopefully) simplify the equation. So long as we're careful to ensure our manipulations are valid (e.g., that we never divide by zero, take the logarithm of something nonpositive, etc.) and check that our solutions make sense, we can solve many differential equations.

Challenges: The following differential equations all have the form $[f(y)]' = g(t)$, which is integrable, although various derivative rules have been applied. See if you can solve them (you may need integrating factors, variable substitutions, and to recognize multiple rules):

(a) $y' = -4y - y^2$

(b) $ty' + y = \frac{1}{y}$

(c) $yy' - \frac{1}{2}y^2\tan(t) = \frac{\sec(t)/2}{1+t}$

These are *significantly* harder than anything I expect to appear on a quiz or exam—they're meant to be hard enough that if you can do them, you have probably mastered the material, but you shouldn't feel like you have to be able to solve them to do well in the class!