Math 4B
Worksheet 2

First Order Equations

The Punch Line: When we have only a single derivative, we can often make progress by recognizing the results of various derivative rules.

The Product Rule and Integrating Factors: Recall that if we have two functions f and g of the same variable x, then the derivative $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$. So, if we recognize the form on the right, we can put it back into the form on the left.

The differential equation $u'(t) = g(t)$ is "easy" to solve: integrating both sides with respect to t (starting at some t_0) gives $u(t) = u(t_0) + \int_{t_0}^{t} g(s)\,ds$ (using s as a "dummy variable" for integration). What we want is to create this situation in an equation we are given. If the equation has (or can be put into) the form

$$y' + p(t)y = g(t),$$

then the left hand side is almost the result of a product rule: it is the result of $[\mu y]'$, where $\mu(t) = e^{\int p(s)\,ds}$ (the "integral" represents any antiderivative), divided by $\mu(t)$ (which we can check is never zero). So if we multiply both sides by $\mu(t)$, we can undo the product rule and integrate, solving the equation!

1 (Integrating Factor): Can an integrating factor be used to create a product rule and solve these differential equations? If so, what is the integrating factor and solution (it's okay to leave it in integral form); if not, what is the obstruction?

(a) $y' + y = \sin(\pi t)$
(b) $xy' + y = \frac{1}{y}$
(c) $xy' + y = e^{-2x}$
(d) $u^2 + t^2 = u'$
(e) $\sqrt{u' + \sin(t)u} = t$
(f) $2uu' + u^2 = \frac{1}{3}t^3 - 1$
(g) $y'' + y = t\cos t$
(h) $\frac{y' - e^t y}{t - e^{-t}} = 1$
(i) $\cos(t)y' + y = 0$

(a) Yes—the integrating factor $\mu(t) = e^t$ makes this equation $\left[y(t)e^t\right]' = e^t\sin(\pi t)$, and integrating both sides yields $ye^t = \int e^s \sin(\pi s)\,ds = \frac{e^t(\sin(\pi t) - \pi\cos(\pi t))}{1+\pi^2} + C$, or $y(t) = Ce^{-t} + \frac{\sin(\pi t) - \pi\cos(\pi t)}{1+\pi^2}$.

(b) No—the $\frac{1}{y}$ term is an issue: the form we seek is a linear equation, so we won't be able to deal with the nonlinearity here (but see the challenge problems later and ask me if you're interested in solving this).

(c) Yes—dividing through by x yields $y' + \frac{1}{x}y = \frac{e^{-2x}}{x}$, so an integrating factor of $\mu(x) = e^{\int \frac{dx}{x}} = x$ gives us what we seek: $[xy]' = e^{-2x}$, so $xy = e^{-2x} + C$, or $y = \frac{1}{x}e^{-2x} + Cx^{-1}$. Note that we must have $x \neq 0$ here, so our solutions are only valid on part of the real line—which one depends on the initial data we're given.

(d) No—again, we have a nonlinearity (the u^2) which will trouble us if we try to write this as a product rule.

(e) Yes—f we square both sides of this equation, we get $u' + \sin(t)u = t^2$, which we can solve with the integrating factor of the form $e^{\int \sin(t)}$, in particular $e^{-\cos(t)}$. Then $y = e^{\cos(t)}\int t^2 e^{-\cos(t)}\,dt + C$ (which can't be solved in terms of elementary functions). Of course, we'll be interested in the domain on which we can solve this; the sign of t determines which square root we need, and we need $u' + \sin(t)u \geq 0$ for the square root to make sense over the real numbers (we can check this against the form we just derived, and verify it is indeed always true).

(f) No—we again have a nonlinear term (u^2 and uu' are both an issue).

(g) No—here we have a second order term, but we're only equipped to handle first order equations with this technique.

(h) Yes—we can rewrite this as $y' - e^t y = t - e^{-t}$. Then, with an integrating factor of $\mu(t) = e^{\int (-e^t)\,dt}$, for example $\mu(t) = e^{-e^t}$, we get $ye^{-e^t} = \int \left(t - e^{-t}\right) e^{-e^t}\,dt + C$. We can write $y = Ce^{e^t} + e^{e^t} \int \left(t - e^{-t}\right) e^{-e^t}\,dt$, and decline to attempt the integral (it's worth noting that in practice, we'd simply approximate it numerically anyway, so it shouldn't be too concerning that this doesn't have a nice form).

(i) Yes—we rewrite as $y' + \sec(t) y = 0$, and put $\mu(t) = e^{\int \sec t\,dt}$ (an example is $\mu(t) = \sec(t) + \tan(t)$, but this is a highly non-trivial integral; don't worry if you don't see how to do it). Then $y(t) = \frac{C}{\mu(t)}$ (the integral of zero is zero, so we just get the "$+C$" term). It's worth noting that this solution doesn't exist for all time (the secant blows up periodically)—the domain of definition depends on initial data.

The Chain Rule and Separation of Variables: Recall that if we have two functions f and g, then the derivative $[f(g(x))]' = f'(g(x))g'(x)$. We're again going to look for the form on the right, to work backwards to the form on the left.

If an equation has the form

$$M(x) + N(y)y' = 0,$$

we pretty much have the result of a chain rule: the $N(y)y'$ term is the chain-rule derivative of an antiderivative of N. So, we can re-write the equation as $N(y)y' = -M(x)$, integrate both sides, and use a substitution on the left integral (namely, $u = y$, $du = y'dx$) to get an explicit answer.

2 (Separation of Variables): Are the following equations separable? If so, what are M and N, and what is the solution? If not, what is the obstruction?

(a) $y' + y = \sin(\pi t)$

(b) $yy' = x$

(c) $xy' = y(1-x)$

(d) $\cos(y)y' = 3t^2$

(e) $y' + x = y + 1$

(f) $\frac{u'}{u-1} = 1$

(a) No—we can't write this as the result of a chain rule, the y term is not with the y' term (so, separation of variables solves different problems than integrating factor, not necessarily harder or easier).

(b) Yes—we can put $N(y) = y$, $M(x) = -x$ (because it's on the other side of the equals sign), and solve this as $[y^2]' = x$, or $y^2 = \frac{1}{2}x^2 + C$, or $y = \pm\sqrt{12x^2 + C}$ (with initial data determining which square root we want). Note that if $C < 0$, this solution is only valid for certain x (in particular, $|x| > \sqrt{-2C}$).

(c) Yes—we can put $N(y) = \frac{1}{y}$, $M(x) = -\frac{1-x}{x}$, and solve this as $\left[\ln\left|y\right|\right]' = \frac{1-x}{x}$, or $\ln\left|y\right| = \ln|x| - x + C$, or $y = cxe^{-x}$ for $c = \pm e^C$.

(d) Yes—we put $N(y) = \cos(y)$, $M(t) = -3t^2$, and solve to get $[-\sin(y)]' = 3t^2$, or $-\sin(y) = t^3 + C$, or (recalling that $-\sin(y) = \sin(-y)$) $y = -\arcsin(t^3 + C)$. Here, of course, we must ensure that $\left|t^3 + C\right| \leq 1$ for this to be well-defined (this will give us information about the domain of definition based on initial data).

(e) No—we can't separate the function into one term purely involving y and one involving x; the separate y is the issue.

(f) Yes—we can put $N(u) = \frac{1}{u-1}$ and $M(x) = -1$, and get $[\ln|u-1|]' = 1$, or $\ln|u-1| = x + C$, or $u(x) = 1 + ce^x$, where $c = e^C$.

Challenge Problems: Of course, both of these methods are just recognizing the result of certain derivative rules. In general, if we're able to manipulate a differential equation in such a way that we recognize the result of a derivative rule, we can "undo" that rule and (hopefully) simplify the equation. So long as we're careful to ensure our manipulations are valid (e.g., that we never divide by zero, take the logarithm of something nonpositive, etc.) and check that our solutions make sense, we can solve many differential equations.

Challenges: The following differential equations all have the form $[f(y)]' = g(t)$, which is integrable, although various derivative rules have been applied. See if you can solve them (you may need integrating factors, variable substitutions, and to recognize multiple rules):

(a) $y' = -4y - y^2$ (b) $ty' + y = \frac{1}{y}$ (c) $yy' - \frac{1}{2}y^2\tan(t) = \frac{\sec(t)/2}{1+t}$

These are *significantly* harder than anything I expect to appear on a quiz or exam—they're meant to be hard enough that if you can do them, you have probably mastered the material, but you shouldn't feel like you have to be able to solve them to do well in the class!

(a) Here we observe that we can rearrange the equation into

$$\frac{y' + 4y}{y^2} = -1.$$

Then, if $\mu(t) = e^{-4t}$, we observe that this is precisely

$$\left[-\frac{\mu}{y}\right]' = -1,$$

an application of the quotient rule. Thus,

$$y(t) = \frac{t + C}{\mu(t)} = te^{4t} + Ce^{4t}.$$

Here, morally, we've just used the integrating factor method, although to create a quotient rule application rather than product rule.

(b) Here we re-write the equation as

$$tyy' + y^2 = 1.$$

We recognize that $tyy' = \frac{1}{2}t[y^2]'$, so putting $u = y^2$ and rearranging we get

$$u' + 2u = 2,$$

which is just a (normal) integrating factor problem with $\mu = e^{2t}$. That is,

$$u(t) = 1 + Ce^{-2t}.$$

But $u = y^2$, so this means

$$y(t) = \pm\sqrt{1 + Ce^{-2t}}.$$

(c) Here, we first observe that we can write this equation as

$$2yy'\cos(t) - y^2\sin(t) = \frac{1}{1+t}.$$

Then we observe that we can (again) use the substitution $u = y^2$ to simplify this to

$$u'\cos(t) - u\sin(t) = \frac{1}{1+t}.$$

Here, we observe that $[\cos(t)]' = -\sin(t)$, so the left hand side is the result of a product rule, so

$$[u\cos(t)]' = \frac{1}{1+t}.$$

Integrating, we get

$$u\cos(t) = C + \ln|1+t|.$$

Dividing by cosine and undoing our substitution, we get

$$y(t) = \pm\sqrt{C\sec(t) + \frac{\ln|1+t|}{\cos(t)}}.$$

Of course, this solution only works for certain values of t, dependent on our initial data.