# Exact Equations

**The Punch Line:** Equations which result from the application of a *multivariable* chain rule can also be solved.

If we have some function $\psi(x, y)$ involving an independent variable $x$ and a dependent variable $y = y(x)$, then the equation

$$\frac{d}{dx}\psi(x, y) = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx} = 0$$

can be integrated (with respect to $x$) to get $\psi(x, y) = c$, which implicitly defines $y(x)$.

We can tell an equation has this form if it can be written

$$M(x, y) + N(x, y)y' = 0$$

and $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ (this is checking $M$ and $N$ are the partial derivatives of the same function $\psi$). In that case, we can get $\psi$ by integrating $M$ with respect to $x$—this will give an answer in terms of some function $h(y)$, which we can solve for by taking the partial derivative with respect to $y$ and setting it equal to $N$.

**1:** Are the following equations exact? If so, what are their solutions (it's okay to leave them implicitly defined—finding $\psi$ is enough)?

(a) $x + yy' = 0$

(b) $2x + y + x^2y' = 0$

(c) $x^2 + \left(y^2 + \sin(x)\sin(y)\right)y' = \cos(x)\cos(y)$

(d) $\frac{xy(1+y')}{2\sqrt{x+y}} + (y + xy')\sqrt{x+y} = 0$

(e) $y^xy' + x^y = 0$

(f) $(y + y')e^x + (1 + y' + xy')e^y = -3y^2y'$

**2:** Write a differential equation and initial condition for the following situation, and solve it.

A family of curves in the plane have the property that at every point $(y, x)$, the slope of the curve is precisely the ratio between the square of the difference between the two coordinates and that quantity minus $y^2$. What is the equation for the curve in this family passing through the point $(3, 2)$?

**Challenge Problems:** Although it won't be on any exam in this course, the textbook explores differential equations which are not exact, but may be made so by multiplying by an appropriate integrating factor. This is similar to the case with linear first-order equations, but the equation determining the integrating factor is in general a partial differential equation. The theory of these is beyond the scope of this class, but in some instances it's clear what the equation has to be (remember that we just need *an* integrating factor that works, not a general one).

**Challenges:** What is the (implicit) solution to the differential equation

$$1 + \frac{x}{y} + \frac{y}{x} + \left(\frac{x}{2y} + 2 + \frac{y}{2x}\right)y' = 0?$$

This is again *significantly* harder than anything that will be on a quiz or exam. I've just included it in case you want a chance to deal with a hard problem—even setting up the equation for the integrating factor takes some work! As ever, I'm more than happy to talk about this in office hours, though.