Exact Equations

The Punch Line: Equations which result from the application of a *multivariable* chain rule can also be solved.

If we have some function $\psi(x,y)$ involving an independent variable x and a dependent variable $y = y(x)$, then the equation

$$\frac{d}{dx}\psi(x,y) = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx} = 0$$

can be integrated (with respect to x) to get $\psi(x,y) = c$, which implicitly defines $y(x)$.

We can tell an equation has this form if it can be written

$$M(x,y) + N(x,y)y' = 0$$

and $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ (this is checking M and N are the partial derivatives of the same function ψ). In that case, we can get ψ by integrating M with respect to x—this will give an answer in terms of some function $h(y)$, which we can solve for by taking the partial derivative with respect to y and setting it equal to N.

1: Are the following equations exact? If so, what are their solutions (it's okay to leave them implicitly defined—finding ψ is enough)?

(a) $x + yy' = 0$

(b) $2x + y + x^2y' = 0$

(c) $x^2 + \left(y^2 + \sin(x)\sin(y)\right)y' = \cos(x)\cos(y)$

(d) $\frac{xy(1+y')}{2\sqrt{x+y}} + (y + xy')\sqrt{x+y} = 0$

(e) $y^xy' + x^y = 0$

(f) $(y + y')e^x + (1 + y' + xy')e^y = -3y^2y'$

(a) Yes—putting $M(x,y) = x$ and $N(x,y) = y$, we see that $M_y = 0 = N_x$. The equation is in fact separable, but to demonstrate how exact equations are solved, we can compute the integral $\int M\,dx = \frac{1}{2}x^2 + h(y) = \psi(x,y)$ (the $h(y)$ term is analogous to the usual "$+C$," but since we're thinking of multivariable functions, our "constant" is allowed to depend on the variable we're not integrating over). Then $\frac{\partial\psi}{\partial y} = h'(y)$, which we set equal to $N(x,y) = y$, so $h(y) = \frac{1}{2}y^2$ (we don't usually write a "$+C$" here, because when we take the derivative of ψ, any constants will not contribute—we just need *an* expression that works, and so are free to choose $C = 0$).

So, our final expression is $\psi(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$. Then solutions to this differential equation are defined by $\psi(x,y) = c$—they'll be arcs of circles in this case (as $x^2 + y^2 = 2c$ is the equation of a circle).

(b) No—putting $M(x,y) = 2x + y$ and $N(x,y) = x^2$, we get $M_y = 1$ but $N_x = 2x$, so $M_y \neq N_x$ and the equation is not exact.

(c) Yes—putting $M(x,y) = x^2 - \cos(x)\cos(y)$ and $N(x,y) = y^2 + \sin(x)\sin(y)$, we get $M_y = \cos(x)\sin(y) = N_x$. Integrating $M(x,y)$ with respect to x gives $\psi(x,y) = \frac{1}{3}x^3 - \sin(x)\cos(y) + h(y)$. Then we get $\psi_y = \sin(x)\sin(y) + h'(y)$, so $h'(y) = y^2$, and $h(y) = \frac{1}{3}y^3$. So, our final expression is $\psi(x,y) = \frac{1}{3}x^3 + \frac{1}{3}y^3 - \sin(x)\cos(y)$.

(d) Yes—putting $M(x,y) = \frac{xy}{2\sqrt{x+y}} + y\sqrt{x+y}$ and $N(x,y) = \frac{xy}{2\sqrt{x+y}} + x\sqrt{x+y}$, we get

$$M_y = \frac{-xy}{4(x+y)^{3/2}} + \frac{x}{2\sqrt{x+y}} + \frac{y}{2\sqrt{x+y}} + \sqrt{x+y} = N_x.$$

Integrating M with respect to x gives

$$\psi(x,y) = \frac{y}{2}\int \frac{x\,dx}{\sqrt{x+y}} + y\int \sqrt{x+y}\,dx = xy\sqrt{x+y} - \frac{2}{3}y(x+y)^{3/2} + \frac{2}{3}y(x+y)^{3/2} + h(y) = xy\sqrt{x+y} + h(y)$$

(use integration by parts for the first integral, and the substitution $u = x + y$ in both, remembering we're treating y as constant as we integrate in x). Then $\psi_y = \frac{xy}{2\sqrt{x+y}} + x\sqrt{x+y} + h'(y)$, so $h'(y) = 0$ and we may take $h(y) = 0$. So, our final expression is $\psi(x, y) = xy\sqrt{x+y}$.

(e) No—putting $M(x, y) = x^y$ and $N(x, y) = y^x$, we see that $M_y = x^y \ln(x)$ and $N_x = y^x \ln(y)$, which are unequal, so the equation is inexact.

(f) Yes—putting $M(x, y) = ye^x + e^y$ and $N(x, y) = e^x + (1+x)e^y + 3y^2$, we see that $M_y = e^x + e^y = N_x$, so the equation is exact. Integrating M with respect to x gives $\psi(x, y) = ye^x + xe^y + h(y)$. Then $\psi_y = e^x + xe^y + h'(y)$, so $h'(y) = e^y + 3y^2$, or $h(y) = e^y + y^3$. Thus, our final expression is $\psi(x, y) = ye^x + (1+x)e^y + y^3$.

2: Write a differential equation and initial condition for the following situation, and solve it.

A family of curves in the plane have the property that at every point (y, x), the slope of the curve is precisely the ratio between the square of the difference between the two coordinates and that quantity minus y^2. What is the equation for the curve in this family passing through the point $(3, 2)$?

The family of curves all satisfy the differential equation

$$y' = \frac{(y-x)^2}{(y-x)^2 - y^2},$$

which we can write in standard form as

$$(x^2 - 2xy)y' - (y-x)^2 = 0.$$

We have the initial condition $y(3) = 2$, as the curve has to pass through $(3, 2)$.

Now, the equation is clearly nonlinear due to the y^2 term that we get from expanding $(y-x)^2$, and it is not separable. However, writing $M(x,y) = -(y-x)^2 = 2xy - x^2 - y^2$ and $N(x,y) = x^2 - 2xy$, we see that $M_y = 2x - 2y = N_x$, so the equation is exact. Integrating M with respect to x gives $\psi(x,y) = x^2y - \frac{1}{3}x^3 - xy^2 + h(y)$, and taking the partial derivative with respect to y gives $\psi_y = x^2 - 2xy + h'(y) = N(x,y)$. Thus, $h'(y) = 0$, so we can take $h(y) = 0$, to get

$$\psi(x,y) = x^2y - \frac{1}{3}x^3 - xy^2 = c$$

for c some constant as the family of equations describing these curves.

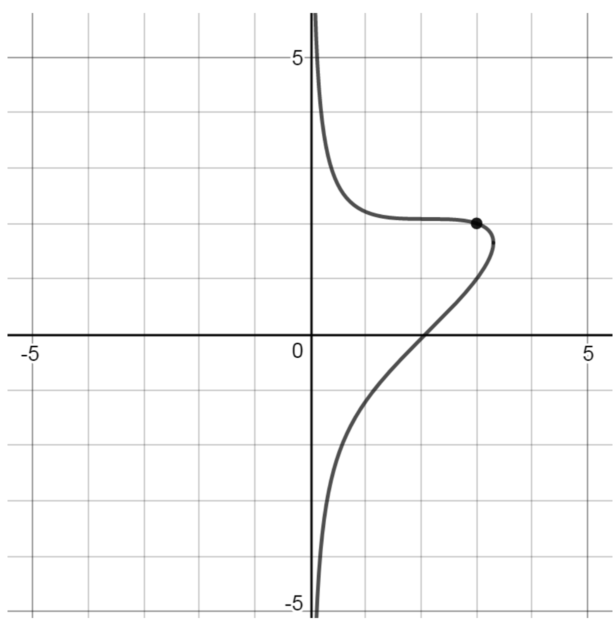
We can plug the point $(3, 2)$ into this to find the particular value of c for the curve we want; this is

$$c = (3)^2(2) - \frac{1}{3}(3)^3 - (3)(2)^2 = -3.$$

Thus, the equation

$$x^2y - \frac{1}{3}x^3 - xy^2 = -3$$

implicitly defines the curve we're looking for. A plot of this is below; note that the curve is *not* a simple one we'd recognize, and in fact is not even a function—we'd have a difficult time computing it through other means.



Challenge Problems: Although it won't be on any exam in this course, the textbook explores differential equations which are not exact, but may be made so by multiplying by an appropriate integrating factor. This is similar to the case with linear first-order equations, but the equation determining the integrating factor is in general a partial differential equation. The theory of these is beyond the scope of this class, but in some instances it's clear what the equation has to be (remember that we just need *an* integrating factor that works, not a general one).

Challenges: What is the (implicit) solution to the differential equation

$$1+\frac{x}{y}+\frac{y}{x}+\left(\frac{x}{2y}+2+\frac{y}{2x}\right)y'=0?$$

This is again *significantly* harder than anything that will be on a quiz or exam. I've just included it in case you want a chance to deal with a hard problem—even setting up the equation for the integrating factor takes some work! As ever, I'm more than happy to talk about this in office hours, though.

Here, we can put $M(x,y)=1+\frac{x}{y}+\frac{y}{x}$ and $N(x,y)=\frac{x}{2y}+2+\frac{y}{2x}$. Then $M_y=-\frac{x}{y^2}+\frac{1}{x}$ and $N_x=\frac{1}{2y}-\frac{y}{2x^2}$, which are not equal. If we multiply the entire equation by some function $\mu(x,y)$, we see that the new "M" (we can call it $\tilde{M}$) satisfies $\tilde{M}_y=-\frac{x\mu}{y^2}+\frac{\mu}{x}+\left(1+\frac{x}{y}+\frac{y}{x}\right)\mu_y$, and a similarly defined $\tilde{N}_x=\frac{\mu}{2y}-\frac{y\mu}{2x^2}+\left(\frac{x}{2y}+2+\frac{y}{2x}\right)\mu_x$. For this to be exact, we need μ to satisfy the partial differential equation

$$-\frac{x\mu}{y^2}+\frac{\mu}{x}+\left(1+\frac{x}{y}+\frac{y}{x}\right)\mu_y=\frac{\mu}{2y}-\frac{y\mu}{2x^2}+\left(\frac{x}{2y}+2+\frac{y}{2x}\right)\mu_x.$$

Rewriting this, we need

$$\left(\frac{x^2+xy+y^2}{xy}\right)\mu_y-\left(\frac{x^2+4xy+y^2}{2xy}\right)\mu_x=\left(\frac{x^2y-y^3-2xy^2+2x^3}{2x^2y^2}\right)\mu$$
$$\left(2x^3y+2x^2y^2+2xy^3\right)\mu_y-\left(x^3y+4x^2y^2+xy^3\right)\mu_x=\left(2x^3+x^2y-2xy^2-y^3\right)\mu.$$

In general, we don't have the techniques to solve this yet. However, note that every coefficient on either side is a *homogeneous* polynomial in x and y—the overall degree of each term is 4 on the left hand side and 3 on the right (this should make sense—taking derivatives reduces the degree of a polynomial, so μ should contribute one degree higher of polynomial than either derivative of it).

We may as well guess that μ is a homogeneous polynomial of x and y as well (we need the two sides to be equal, and there is nothing to break the homogeneity), and a first start could be to try $\mu=x^ny^m$ (a single homogeneous term). Then $\mu_x=nx^{n-1}y^m$ and $\mu_y=mx^ny^{m-1}$. Plugging this in gives the polynomial equation

$$2mx^{n+3}y^m+(2m-n)x^{n+2}y^{m+1}+(2m-4n)x^{n+1}y^{m+2}-nx^ny^{m+3}=2x^{n+3}y^m+x^{m+2}y^{m+1}-2x^{m+1}y^{n+2}-x^ny^{m+3}.$$

Setting the coefficients of like terms equal, we see that we need $n=1$ (from the last term), $m=1$ (from the first term), and that this works (the middle two terms work out for this choice of m and n). Thus, $\mu(x,y)=xy$ is an integrating factor which works.

So, we use the equation

$$xy+x^2+y^2+\left(\frac{x^2}{2}+2xy+\frac{y^2}{2}\right)y'=0,$$

which is exact (you can check this). Then the x integral of the new M is $\frac{1}{2}x^2y+\frac{1}{3}x^3+xy^2+h(y)$, which has y derivative $\frac{1}{2}x^2+2xy+h'(y)$, so $h'(y)=\frac{1}{2}y^2$, so $h(y)=\frac{1}{6}y^3$. Then we get an (implicit) solution

$$\frac{1}{2}x^2y+\frac{1}{3}x^3+xy^2+\frac{1}{6}y^3=c.$$