Existence and Uniqueness Theory

The Punch Line: We can determine if it's possible to solve IVPs without actually solving them.

Linear Theory: If we have a differential equation $y' + p(t)y = g(t)$ and initial condition $y(t_0) = y + 0$, and p and g are both continuous on the same open interval (α, β) containing t_0, then there is a unique function $\phi(t)$ that satisfies the initial value problem. So, if we can find all of the intervals on which p and g are simultaneously continuous, we can find intervals on which we are guaranteed *some* solution to IVPs (without actually solving the equation). This is useful if, for example, we want to be sure that approximate numerical solutions are close to some real solution.

1: For the following linear equations, find the intervals (in t) on which we are guaranteed unique solutions to an IVP with the following differential equations:

(a) $y' + ay = 0$

(b) $ty' + 2y = e^{-t}$

(c) $y' + \frac{1}{t}y = \frac{1}{1-t}$

(d) $[y \ln t]' = 0$

(e) $y' + \frac{t+2}{t^2-9}y = \frac{t^2+1}{t^2-4}$

(f) $t(t-2)y' + (t-1)y = \ln\left(t^2 - 16\right)$

General First-Order Theory: If we have a differential equation $y' = f(t,y)$ and $f(t,y)$ is continuous on some rectangle $t \in (\alpha,\beta)$, $y \in (\gamma,\delta)$ containing the point (t_0,y_0) (our initial data), then there is some interval (t_0-h,t_0+h) contained in (α,β) where we have a solution to the IVP, $\phi(t)$. If $\frac{\partial f}{\partial y}$ is also continuous on the rectangle, then that solution is unique. It's worth noting that the interval on which we have (unique) solutions are in general *not* the full interval on which f is continuous, and may be either larger or smaller: we are just guaranteed the existence of *some* interval that works—finding what it is generally involves finding the solution to the DE itself.

2: For the following first-order equations, find the initial conditions (t_0,y_0) that do *not* result in guaranteed solutions to the DE, and do not result in guaranteed unique solutions.

(a) $y' = \left|y\right|$

(b) $y' + ty = y^2$

(c) $yy' = \frac{1}{2t}$

(d) $(x+t) + \left(t - x^2\right)x' = 0$