Existence and Uniqueness Theory

The Punch Line: We can determine if it's possible to solve IVPs without actually solving them.

Linear Theory: If we have a differential equation $y' + p(t)y = g(t)$ and initial condition $y(t_0) = y + 0$, and p and g are both continuous on the same open interval (α, β) containing t_0, then there is a unique function $\phi(t)$ that satisfies the initial value problem. So, if we can find all of the intervals on which p and g are simultaneously continuous, we can find intervals on which we are guaranteed *some* solution to IVPs (without actually solving the equation). This is useful if, for example, we want to be sure that approximate numerical solutions are close to some real solution.

1: For the following linear equations, find the intervals (in t) on which we are guaranteed unique solutions to an IVP with the following differential equations:

(a) $y' + ay = 0$

(b) $ty' + 2y = e^{-t}$

(c) $y' + \frac{1}{t}y = \frac{1}{1-t}$

(d) $[y \ln t]' = 0$

(e) $y' + \frac{t+2}{t^2-9}y = \frac{t^2+1}{t^2-4}$

(f) $t(t-2)y' + (t-1)y = \ln\left(t^2 - 16\right)$

(a) Here we get $p(t) = a$ and $g(t) = 0$, both of which are continuous functions for all t. Thus, given *any* initial data, we have a unique solution for all time t.

(b) Here, we get $p(t) = \frac{2}{t}$ and $g(t) = t^{-1}e^{-t}$, which both have singularities at $t = 0$ and nowhere else. Thus, we have the two possible intervals $t \in (-\infty, 0)$ and $t \in (0, \infty)$ on which our functions could be defined. If t_0 is positive (or negative), we will have a unique solution for all positive (negative) time t.

(c) Here, we get $p(t) = \frac{1}{t}$ and $g(t) = \frac{1}{1-t}$, so the two "bad points" are $t = 0$ and $t = 1$. Thus, our intervals are $(-\infty, 0)$, $(0, 1)$, and $(1, \infty)$.

(d) Here, we first expand the DE to $y' \ln t + \frac{1}{t}y = 0$, or $y' + \frac{1}{t \ln t} = 0$. Here, we have problems if $t \leq 0$, so our possible interval is $t \in (0, \infty)$.

(e) Here, we get problems at $t = \pm 2, \pm 3$, so our intervals are $(-\infty, -3)$, $(-3, -2)$, $(-2, 2)$, $(2, 3)$, and $(3, \infty)$. It's worth noting that we need to consider the "bad points" of *both* p and g at the same time, as some of these intervals have one endpoint determined by p and the other by g.

(f) Here we get problems at $t = 0, 2, \pm 4$, so our intervals are $(-\infty, -4)$, $(-4, 0)$, $(0, 2)$, $(2, 4)$, and $(4, \infty)$.

General First-Order Theory: If we have a differential equation $y' = f(t,y)$ and $f(t,y)$ is continuous on some rectangle $t \in (\alpha, \beta)$, $y \in (\gamma, \delta)$ containing the point (t_0, y_0) (our initial data), then there is some interval $(t_0 - h, t_0 + h)$ contained in (α, β) where we have a solution to the IVP, $\phi(t)$. If $\frac{\partial f}{\partial y}$ is also continuous on the rectangle, then that solution is unique. It's worth noting that the interval on which we have (unique) solutions are in general *not* the full interval on which f is continuous, and may be either larger or smaller: we are just guaranteed the existence of *some* interval that works—finding what it is generally involves finding the solution to the DE itself.

2: For the following first-order equations, find the initial conditions (t_0, y_0) that do *not* result in guaranteed solutions to the DE, and do not result in guaranteed unique solutions.

(a) $y' = |y|$

(b) $y' + ty = y^2$

(c) $yy' = \frac{1}{2t}$

(d) $(x + t) + \left(t - x^2\right)x' = 0$

(a) Here, we are guaranteed a solution for any (t_0, y_0), but are not guaranteed uniqueness if $y_0 = 0$, as the derivative is discontinuous (in fact, not defined) there.

(b) Here $f(t,y) = y(t - y)$, and $f_y = t - 2y$, which are both always continuous, so we are always guaranteed a unique solution for some interval.

(c) Here, $f(t,y) = \frac{1}{2ty}$, with $f_y = \frac{-1}{2ty^2}$, so we need for $y_0 \neq 0$ *and* $t_0 \neq 0$ for both existence and uniqueness.

(d) Here $f(t,x) = \frac{1}{x-t}$, and $f_x = \frac{1}{t-x^2}$, so we will only have existence problems with this theory if $x_0 = t_0$, and only uniqueness problems if $x_0^2 = t_0$ (or of course $x_0 = t_0$). In fact, we can solve this exact equation, and as it turns out for some pairs with $x_0 = t_0$ we *will* in fact have solutions: the theory simply fails to guarantee them, rather than proving they don't exist.