Math 4B
Worksheet 4b

Second Order Equations I

The Punch Line: The solutions to many second-order constant-coefficient equations may be deduced by solving quadratic equations.

Linear Homogeneous Constant-Coefficient Second Order Equations: If we have the differential equation $ay'' + by' + cy = 0$, we can ask ourselves if there are any solutions of the form $y(t) = e^{rt}$. If so, we would have $ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$, which can only happen if $ar^2 + br + c = 0$ ($e^{rt} > 0$ for all t). This quadratic equation is the *characteristic equation*, and its roots tell us what solutions to expect. In general, the solution to an IVP is a linear combination of the two possibilities (if there are two distinct roots), with coefficients chosen to match the initial conditions.

1: Solve the following differential equations with the initial conditions $y(0) = 1$ and $y'(0) = 0$:

(a) $y'' - 9y = 0$

(b) $y'' - 3y' + 2y = 0$

(c) $2y'' + 3y' + y = 0$

(d) $y'' + y' = 0$

(a) The characteristic equation here is $r^2 - 9 = 0$, so $r = \pm 3$, so our solution is of the form $C_1e^{3t} + C_2e^{-3t}$. Our initial conditions give $C_1 + C_2 = 1$ and $3(C_1 - C_2) = 0$, so $C_1 = C_2 = \frac{1}{2}$. Thus, our final answer is $y(t) = \frac{1}{2}e^{3t} + \frac{1}{2}e^{-3t}$.

(b) The characteristic equation here is $r^2 - 3r + 2 = (r-1)(r-2) = 0$, so our general solution has the form $C_1e^t + C_2e^{2t}$. The first initial condition gives $C_1 + C_2 = 1$, the second gives $C_1 + 2C_2 = 0$, or $C_1 = -2C_2$, so $C_2 = -1$ and $C_1 = 2$ for a final solution of $y(t) = 2e^t - e^{2t}$.

(c) The characteristic equation here is $2r^2 + 3r + 1 = 2(r + \frac{1}{2})(r + 1) = 0$, so our general solution is $C_1e^t + C_2e^{t/2}$. Initial conditions give $C_1 + C_2 = 1$ and $C_1 + \frac{1}{2}C_2 = 0$, so $C_2 = -2C_1$, $C_1 = -1$, so our final answer is $2e^{t/2} - e^t$.

(d) The characteristic equation here is $r^2 + r = r(r+1) = 0$, so our general solution is $C_1 + C_2e^{-t}$. Our second initial condition gives $C_2 = 0$, so our final answer is $y(t) = 1$.

Challenge Problems: Sometimes, the solution technique above will fail (e.g., what happens if the characteristic equation has only one root?). Also, we are occasionally interested in *boundary value problems*, where instead of a derivative condition we have a value condition at a different point. The ideas used in dealing with these situations are similar (although we haven't done them yet in class, so don't worry if you can't solve these problems).

> **Challenges:** Solve these differential equations
>
> (a) $y'' - 4y' + 4y = 0$, where $y(0) = 1$, $y'(0) = 0$ (hint: think about $y'' = 0$ first)
>
> (b) $y'' - y = 0$, where $y(0) = 1$ and $y(1) = 0$
>
> (c) $y'' + y = 0$, where $y(0) = y(\pi) = 0$

(a) Here, our characteristic equation is $r^2 - 4r - 4 = (r-2)^2 = 0$. So, we expect e^{2t} to be a solution. However, if $y(t) = Ce^{2t}$, then $y'(t) = 2Ce^{2t}$; we can't satisfy both initial conditions with this function.

As it turns out (more on this later, or ask me!), the function $y(t) = te^{2t}$ also satisfies the differential equation (try it), so our general solution is $C_1e^{2t} + C_2te^{2t}$. Then $y' = (2C_1 + C_2)e^{2t} + 2C_2te^{2t}$. Our first initial condition gives $C_1 = 1$, and our second gives $C_2 = -2$, for a final answer of $y(t) = e^{2t} - 2te^{2t}$. This phenomenon (a t-multiple of the solution being necessary) is called *resonance*.

(b) The general solution here is $C_1e^t + C_2e^{-t}$. Our first initial condition gives $C_1 + C_2 = 1$, and our second gives $eC_1 + \frac{1}{e}C_2 = 0$. Then $C_2 = -e^2C_1$, and thus $C_1 = \frac{-1}{e^2-1}$ and $C_2 = \frac{e^2}{e^2-1}$, for a final solution of $y(t) = \frac{e^{2-t}-e^t}{e^2-1}$. We can check that this matches the boundary values, as desired.

(c) Our characteristic polynomial here is $r^2 + 1 = 0$. This has complex roots; we could put a general solution as $C_1e^{it} + C_2e^{-it}$, but this is hard to physically interpret. Better is to observe that the equation is $y'' = -y$, and recall that this is true of the sine and cosine functions (which gives us two solutions, as expected).

Putting $y(t) = A\sin(t) + B\cos(t)$, we get $y(0) = B$ and $y(\pi) = -B$. Thus, any solution $y(t) = A\sin(t)$ solves the boundary value problem. Note that this solution is obviously not unique—this is much more common for boundary value problems than initial value problems (where all the information is at a single point). If we want a unique solution, we'll need more data (such as a value of y at some point not of the form $n\pi$).