Second Order Equations II

The Punch Line: Repeated and complex roots require only slight elaborations on the basic solution techniques from last section.

Repeated Roots and Reduction of Order: If the characteristic polynomial of our differential equation has a repeated root (e.g., is of the form $(r-a)^2 = 0$), then a simple exponential solution is insufficient. Instead, we should expect solutions of the form $y(t) = C_1 e^{at} + C_2 t e^{at}$.

More generally, if we know $y_1(t)$ is a solution to a DE, then often we can find a solution $y_2(t) = v(t)y_1(t)$ for some function $v(t)$. Plugging this y_2 into the DE will give a DE for v which is (hopefully) simpler to solve.

1: Solve the following differential equations with given initial conditions:

(a) $y'' - 2y' + y = 0$ with $y(0) = 1$ and $y'(0) = 1$

(b) $y'' + 18y' + 81y = 0$ with $y(0) = 0$ and $y'(0) = 4$

(c) $2t^2y'' - ty' + y = 0$ with $y(1) = 1$ and $y'(1) = 0$ ($y_1(t) = t$ is a solution to the DE)

Complex Roots: If we have complex roots, e.g. $r = \alpha \pm \beta i$, we expect solutions of the form $y(t) = C_1 e^{\alpha t} e^{i\beta t} + C_2 e^{\alpha t} e^{-i\beta t}$. Using Euler's Identity $e^{i\theta} = \cos\theta + i\sin\theta$ shows that this actually has the (real-valued) solutions $y(t) = D_1 e^{\alpha t}\cos(\beta t) + D_2 e^{\alpha t}\sin(\beta t)$, which are often easier to use.

2: Solve these differential equations

(a) $y'' + 4y = 0$ with $y(0) = 1$ and $y'(0) = 2$

(b) $y'' + 2y' + 2y = 0$ with $y(0) = 0$ and $y'(0) = 1$

(c) $y'' + 2y' + 2y = 0$ with $y(\pi) = 1$ and $y'(\pi) = 0$

(d) $y'' + 4y' + 13 = 0$ with $y(0) = 1$ and $y'(0) = 0$