Second Order Equations II

The Punch Line: Repeated and complex roots require only slight elaborations on the basic solution techniques from last section.

Repeated Roots and Reduction of Order: If the characteristic polynomial of our differential equation has a repeated root (e.g., is of the form $(r-a)^2 = 0$), then a simple exponential solution is insufficient. Instead, we should expect solutions of the form $y(t) = C_1 e^{at} + C_2 t e^{at}$.

More generally, if we know $y_1(t)$ is a solution to a DE, then often we can find a solution $y_2(t) = v(t)y_1(t)$ for some function $v(t)$. Plugging this y_2 into the DE will give a DE for v which is (hopefully) simpler to solve.

1: Solve the following differential equations with given initial conditions:

(a) $y'' - 2y' + y = 0$ with $y(0) = 1$ and $y'(0) = 1$

(b) $y'' + 18y' + 81y = 0$ with $y(0) = 0$ and $y'(0) = 4$

(c) $2t^2 y'' - ty' + y = 0$ with $y(1) = 1$ and $y'(1) = 0$ ($y_1(t) = t$ is a solution to the DE)

(a) The characteristic equation here is $r^2 - 2r + 1 = (r-1)^2 = 0$, so we expect a solution of the form $y(t) = C_1 e^t + C_2 t e^t$. Then $C_1 = 1$ (from the $y(0) = 1$ condition), so $y'(t) = e^t + C_2(te^t + e^t)$, so $C_2 = 0$ (from the $y'(0) = 1$ condition), for a final solution of $y(t) = e^t$.

(b) The characteristic equation here is $r^2 + 18r + 81 = (r+9)^2 = 0$, so we expect a solution of the form $y(t) = C_1 e^{9t} + C_2 t e^{9t}$. Then $C_1 = 0$, so $y'(t) = C_2(9te^{9t} + e^{9t})$, so $C_2 = 4$ and our final solution is $y(t) = 4te^{9t}$.

(c) Here, we try a solution of the form $y_2(t) = tv(t)$, with $y_2'(t) = tv'(t) + v(t)$ and $y_2''(t) = tv''(t) + (1+t)v'(t)$. Then we get the differential equation $2t^3 v''(t) + 4t^2 v'(t) - t^2 v'(t) - tv(t) + tv(t) = 2t^3 v''(t) + 3t^2 v'(t) = 0$, or $v''(t) + \frac{1}{2t} v'(t) = 0$.

Here we can essentially use an integrating factor of $\mu(t) = t^{3/2}$ to get $\left[t^{3/2} v'\right]' = 0$, so $v' = \tilde{C} t^{-3/2}$, so $v = Ct^{-1/2} + D$, so ultimately $y(t) = C_1 \sqrt{t} + C_2 t$. Then we get $C_1 + C_2 = 1$ from the initial value condition and $\frac{1}{2} C_1 + C_2 = 0$ from the initial derivative condition. So, we get $C_1 = -2C_2$ and $-C_2 = 1$, so $C_2 = -1$ and $C_1 = 2$, for a final solution of $y(t) = 2\sqrt{t} - t$.

It's worth noting that the Wronskian here is $W(y_1, y_2)(t) = t\left(\frac{1}{2\sqrt{t}}\right) - \sqrt{t} = -\frac{1}{2}\sqrt{t}$, which is nonzero around $t = 1$. We would have problems near zero, which is unsurprising as $p(t) = \frac{-1}{2t}$ is not continuous there.

Complex Roots: If we have complex roots, e.g. $r = \alpha \pm \beta i$, we expect solutions of the form $y(t) = C_1 e^{\alpha t} e^{i\beta t} + C_2 e^{\alpha t} e^{-i\beta t}$. Using Euler's Identity $e^{i\theta} = \cos\theta + i\sin\theta$ shows that this actually has the (real-valued) solutions $y(t) = D_1 e^{\alpha t}\cos(\beta t) + D_2 e^{\alpha t}\sin(\beta t)$, which are often easier to use.

2: Solve these differential equations

(a) $y'' + 4y = 0$ with $y(0) = 1$ and $y'(0) = 2$

(b) $y'' + 2y' + 2y = 0$ with $y(0) = 0$ and $y'(0) = 1$

(c) $y'' + 2y' + 2y = 0$ with $y(\pi) = 1$ and $y'(\pi) = 0$

(d) $y'' + 4y' + 13 = 0$ with $y(0) = 1$ and $y'(0) = 0$

(a) Here, our characteristic equation is $r^2 + 4r = 0$, so the roots are $r = \pm 2i$. Thus, we have a general solution $y(t) = C_1\cos(2t) + C_2\sin(2t)$. Then $C_1 = 1$ and so $y'(t) = -2\sin(2t) + 2C_2\cos(2t)$, so $C_2 = 1$ and our final solution is $y(t) = \cos(2t) + \sin(2t)$.

(b) Here, our characteristic equation is $r^2 + 2r + 2 = 0$, so the quadratic formula gives $r = -1 \pm i$. So, our general solution has the form $y(t) = C_1 e^{-t}\cos(t) + C_2 e^{-t}\sin(t)$. Then $C_1 = 0$ and $y'(t) = -C_2 e^{-t}\sin(t) + C_2 e^{-t}\cos(t)$, so $C_2 = 1$ and $y(t) = e^{-t}\sin(t)$.

(c) Here, we have the same general solution, so get $y(\pi) = -C_1 e^{-\pi}$, or $C_1 = -e^{\pi}$, and

$$y'(t) = e^{\pi}e^{-t}\cos(t) + e^{\pi}e^{-t}\sin(t) - C_2 e^{-t}\sin(t) + C_2 e^{-t}\cos(t),$$

so $y'(\pi) = -1 - C_2 e^{-\pi}$, or $C_2 = -e^{\pi}$. Thus, $y(t) = -e^{\pi - t}(\cos(t) + \sin(t))$.

(d) Here, the characteristic equation is $r^2 + 4r + 13 = 0$, so the quadratic formula gives $r = -2 \pm 3i$. So, our general solution is $y(t) = C_1 e^{-2t}\cos(3t) + C_2 e^{-2t}\sin(3t)$. Then $C_1 = 1$, and

$$y'(t) = -2e^{-2t}\cos(3t) - 3e^{-2t}\sin(3t) - 2C_2 e^{-2t}\sin(3t) + 3C_2 e^{-2t}\cos(3t),$$

so $-2 + 3C_2 = 0$, or $C_2 = \frac{2}{3}$, and thus $y(t) = e^{-2t}\cos(3t) + \frac{2}{3}e^{-2t}\sin(3t)$.