Second Order Equations III

The Punch Line: Non-homogeneous equations can be approached by exploiting the homogeneous equation.

Undetermined Coefficients: Given the equation $y'' + by' + cy = g(t)$, if g is "nice enough" a particular solution can often by found by solving for a y which is a linear combination of g and its derivatives (here "nice enough" essentially means repeated derivatives only result in a finite number of functions with various constants; examples are polynomials, exponentials, and trigonometric functions). If g is a solution to the homogeneous equation, additional factors of t multiplying g and its derivatives may be necessary (for the same reasons they appear in multiple roots).

1: Solve the following differential equations (if there are no initial conditions, give the general solution):

(a) $y'' - y = e^{2t}$ with $y(0) = 1$ and $y'(0) = 0$

(b) $y'' + 5y' + 6y = 1 - t^2$

(c) $y''+2y'+2y = 5\cos(t)+t$ with $y(0) = \frac{9}{2}$, $y'(0) = -\frac{3}{2}$

(d) $y'' + 2y' + y = e^{-t}$

Variation of Parameters: If we have a differential equation $y'' + p(t)y' + q(t)y = g(t)$ and know that y_1, y_2 are a fundamental set of solutions, we can solve the system in one (complicated) step as

$$y(t) = -y_1(t) \int \frac{y_2(t)g(t)}{W(y_1, y_2)(t)}\,dt + y_2(t) \int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)}\,dt.$$

This offloads the difficulty into computing an integral (and the proof, of course).

2: Find the general solutions of these differential equations using variation of parameters (it's for practice, even if undetermined coefficients would be easier; leaving the solutions in integral form is fine here)

(a) $y''(t) + y(t) = t$

(b) $t^2 y''(t) + y(t) = \cos(t)$ where $y_1(t) = t^2$ and $y_2(t) = t^{-1}$ is a fundamental set of solutions