Math 4B
Worksheet 6

Second Order Equations III

The Punch Line: Non-homogeneous equations can be approached by exploiting the homogeneous equation.

Undetermined Coefficients: Given the equation $y'' + by' + cy = g(t)$, if g is "nice enough" a particular solution can often by found by solving for a y which is a linear combination of g and its derivatives (here "nice enough" essentially means repeated derivatives only result in a finite number of functions with various constants; examples are polynomials, exponentials, and trigonometric functions). If g is a solution to the homogeneous equation, additional factors of t multiplying g and its derivatives may be necessary (for the same reasons they appear in multiple roots).

1: Solve the following differential equations (if there are no initial conditions, give the general solution):

(a) $y'' - y = e^{2t}$ with $y(0) = 1$ and $y'(0) = 0$

(b) $y'' + 5y' + 6y = 1 - t^2$

(c) $y''+2y'+2y = 5\cos(t)+t$ with $y(0) = \frac{9}{2}$, $y'(0) = -\frac{3}{2}$

(d) $y'' + 2y' + y = e^{-t}$

(a) The characteristic equation here is $r^2 - 1 = 0$, so $r = \pm 1$. Then the homogeneous solution is $y_h(t) = C_1 e^t + C_2 e^{-t}$. We search for a particular solution of the form $y_p(t) = Ae^{2t}$: this has $y_p''(t) = 4Ae^{2t}$, so our differential equation gives $3Ae^{2t} = e^{2t}$, so $A = \frac{1}{3}$.

Then the general solution is $y(t) = C_1 e^t + C_2 e^{-t} + \frac{1}{3}e^{2t}$. Our first initial condition gives $C_1 + C_2 + \frac{1}{3} = 1$, and our second that $C_1 - C_2 + \frac{2}{3} = 0$. Solving these gives $C_1 = 0$ and $C_2 = \frac{2}{3}$, for a final solution of $y(t) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$.

(b) Here the characteristic equation is $r^2 + 5r + 6 = (r + 2)(r + 3) = 0$, so the homogeneous solution is $y_h(t) = C_1 e^{-2t} + C_2 e^{-3t}$. For the particular solution, we try $y_p(t) = at^2 + bt + c$ (the derivatives of $1 - t^2$ will all be polynomials of degree at most two), with $y_p'(t) = 2at + b$ and $y_p''(t) = 2a$. In the differential equation this gives $2a + 10at + 5b + 6at^2 + 6bt + 6c = 6at^2 + (10a + 6b)t + (2a + 5b + 6c) = 1 - t^2$. Equating like powers (starting with t^2), we see that $a = -\frac{1}{2}$, $b = \frac{5}{6}$, and $c = \frac{-13}{36}$. So, the general solution is $y(t) = C_1 e^{-2t} + C_2 e^{-3t} - \frac{1}{2}t^2 + \frac{5}{6}t - \frac{13}{36}$.

(c) The characteristic equation is $r^2 + 2r + 2 = 0$, which has roots at $r = -1 \pm i$, so the general solution to the homogeneous equation is $y_h(t) = C_1 e^{-t}\cos(t) + C_2 e^{-t}\sin(t)$. We try $y_p(t) = A\cos(t) + B\sin(t) + Ct + D$, with $y_p'(t) = -A\sin(t) + B\cos(t) + C$ and $y_p''(t) = -A\cos(t) - B\sin(t)$; putting these in the DE and solving gives $(A + 2B)\cos(t) + (B - 2A)\sin(t) + 2Ct + (2C + 2D) = 5\cos(t) + t$. Then $B = 2A$ and $A = 1$, while $C = \frac{1}{2}$ and $D = -\frac{1}{2}$, so the general solution is

$$y(t) = C_1 e^{-t}\cos(t) + C_2 e^{-t}\sin(t) + \cos(t) + 2\sin(t) + \frac{1}{2}(t - 1).$$

Our $y(0)$ condition gives $C_1 + \frac{1}{2} = \frac{9}{2}$, so $C_1 = 4$. Then

$$y'(t) = -4e^{-t}\sin(t) - 4e^{-t}\cos(t) + C_2 e^{-t}\cos(t) - C_2 e^{-t}\sin(t) - \sin(t) + 2\cos(t) + \frac{1}{2},$$

so $y'(0) = -4 + C_2 + 2 + \frac{1}{2} = -\frac{3}{2}$, so $C_2 = 1$ and our final answer is

$$y(t) = 4e^{-t}\cos(t) + e^{-t}\sin(t) + \cos(t) + 2\sin(t) + \frac{1}{2}(t - 1).$$

(d) Our differential equation has characteristic equation $r^2 + 2r + 1 = (r + 1)^2 = 0$, so the general homogeneous solution is $C_1 e^{-t} + C_2 te^{-t}$. Our $g(t)$ and $tg(t)$ are in this form, so we try $y_p(t) = At^2 e^{-t}$ (the homogeneous solution will include any e^{-t} or te^{-t} terms). Putting this through our differential equation gives $\left[(At^2 - 4At + 2A) + 2(-At^2 + 2At) + (At^2)\right]e^{-t} = 2Ae^{-t}$, so $A = \frac{1}{2}$. This means the general solution to the DE is $y(t) = C_1 e^{-t} + C_2 te^{-t} + \frac{1}{2}t^2 e^{-t}$.

Variation of Parameters: If we have a differential equation $y'' + p(t)y' + q(t)y = g(t)$ and know that y_1, y_2 are a fundamental set of solutions, we can solve the system in one (complicated) step as

$$y(t) = -y_1(t)\int \frac{y_2(t)g(t)}{W(y_1,y_2)(t)}\,dt + y_2(t)\int \frac{y_1(t)g(t)}{W(y_1,y_2)(t)}\,dt.$$

This offloads the difficulty into computing an integral (and the proof, of course).

2: Find the general solutions of these differential equations using variation of parameters (it's for practice, even if undetermined coefficients would be easier; leaving the solutions in integral form is fine here)

(a) $y''(t) + y(t) = t$

(b) $t^2 y''(t) + y(t) = \cos(t)$, where $y_1(t) = t^2$ and $y_2(t) = t^{-1}$ is a fundamental set of solutions

(a) Here we recognize $y_1(t) = \cos(t)$, $y_2(t) = \sin(t)$ as a fundamental set of solutions (by finding the homogeneous solution). Then $W(y_1,y_2)(t) = \cos^2(t) + \sin^2(t) = 1$, so our solution is

$$y(t) = -\cos(t)\int t\sin(t)\,dt + \sin(t)\int t\cos(t)\,dt.$$

Now, integration by parts gives that $\int t\sin(t)\,dt = -t\cos(t) + \sin(t) + C$ and $\int t\cos(t)\,dt = t\sin(t) + \cos(t) + D$ (where C and D are constants of integration), so our final solution is

$$\begin{aligned} y(t) &= -\cos(t)\left[-t\cos(t) + \sin(t) + C\right] + \sin(t)\left[t\sin(t) + \cos(t) + D\right] \\ &= t\left[\cos^2(t) + \sin^2(t)\right] - C\cos(t) + D\sin(t) \\ &= t - C\cos(t) + D\sin(t), \end{aligned}$$

which is precisely what we would get by using undetermined coefficients.

(b) Here we get $W(y_1,y_2) = t^2(-t^{-2}) - 2t(t^{-1}) = -3$, so our final solution is

$$y(t) = -t^2\int \frac{t^{-1}\cos(t)}{-3}\,dt + t^{-1}\int \frac{t^2\cos(t)}{-3}\,dt.$$

We can compute $\int t^2\cos(t)\,dt$ by parts to get $t^2\sin(t) + t\cos(t) - \sin(t) + C$, but $\int t^{-1}\cos(t)\,dt$ doesn't have a solution in terms of elementary functions. So, the best form we can express the general solution in is

$$y(t) = \frac{t^2}{3}\int t^{-1}\cos(t)\,dt - \frac{t}{3}\sin(t) - \frac{1}{3}\cos(t) + \frac{1}{3t}\sin(t) - \frac{C}{3t}$$

(keeping in mind that the indefinite integral is hiding another constant of integration dependent on initial data).