Math 4B
Worksheet 7a

Higher Order Equations

The Punch Line: Higher order equations behave essentially like second-order equations with more terms.

Setup: Given a differential equation $a_n y^{(n)} + \cdots + a_0 y = g(t)$, we write down the characteristic equation $a_n r^n + \cdots + a_0 = 0$. If $r = \lambda$ is a root with multiplicity k (that is, $(r-\lambda)^k$ is a term in the factorization of the polynomial), then $(C_1 + \cdots + C_k t^{k-1})e^{\lambda t}$ is a term in the general solution. If $r = \alpha \pm i\beta$ is a pair of roots with multiplicity k (that is, a quadratic with those roots appears with multiplicity k in the factorization), then

$$e^{\alpha t}\left(A_1\cos(\mu t) + B_1\sin(\mu t) + A_2 t\cos(\mu t) + B_2 t\sin(\mu t) + \cdots + A_k t^{k-1}\cos(\mu t) + B_k t^{k-1}\sin(\mu t)\right)$$

is a term in the general solution. That is, each solution you'd expect from the characteristic equation is present—there are simply more of them due to the additional roots higher-order equations have.

Undetermined coefficients works the same way as it does for second order equations. Variation of parameters uses a very similar formula: if $W_m(t)$ denotes the Wronskian with the mth column replaced with the vector $\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$, then the solution is

$$y(t) = \sum_{m=1}^{n} y_m(t) \int \frac{g(t)W_m(t)}{W(t)}\,dt.$$

1: For each DE, write down the general solution.

(a) $y''' + 2y'' - y' - 2y = 0$

(b) $y''' + y = 0$

(c) A homogeneous DE with characteristic polynomial $(r+2)(r+4)^2(r^2+2r+2)^2$

(d) $y^{(4)} - y = te^{-t}$