Higher Order Equations

The Punch Line: Higher order equations behave essentially like second-order equations with more terms.

Setup: Given a differential equation $a_n y^{(n)} + \cdots + a_0 y = g(t)$, we write down the characteristic equation $a_n r^n + \cdots + a_0 = 0$. If $r = \lambda$ is a root with multiplicity k (that is, $(r-\lambda)^k$ is a term in the factorization of the polynomial), then $(C_1 + \cdots + C_k t^{k-1})e^{\lambda t}$ is a term in the general solution. If $r = \alpha \pm i\beta$ is a pair of roots with multiplicity k (that is, a quadratic with those roots appears with multiplicity k in the factorization), then

$$e^{\alpha t}\left(A_1\cos(\mu t) + B_1\sin(\mu t) + A_2 t\cos(\mu t) + B_2 t\sin(\mu t) + \cdots + A_k t^{k-1}\cos(\mu t) + B_k t^{k-1}\sin(\mu t)\right)$$

is a term in the general solution. That is, each solution you'd expect from the characteristic equation is present—there are simply more of them due to the additional roots higher-order equations have.

Undetermined coefficients works the same way as it does for second order equations. Variation of parameters uses a very similar formula: if $W_m(t)$ denotes the Wronskian with the mth column replaced with the vector $\begin{bmatrix}0\\ \vdots\\ 0\\ 1\end{bmatrix}$, then the solution is

$$y(t) = \sum_{m=1}^{n} y_m(t)\int \frac{g(t)W_m(t)}{W(t)}\,dt.$$

1: For each DE, write down the general solution.

(a) $y''' + 2y'' - y' - 2y = 0$

(b) $y''' + y = 0$

(c) A homogeneous DE with characteristic polynomial $(r+2)(r+4)^2(r^2+2r+2)^2$

(d) $y^{(4)} - y = te^{-t}$

(a) The characteristic equation here is $r^3 + 2r^2 - r - 2 = 0$. We rewrite this as $r^2(r+2) - (r+2) = (r^2-1)(r+2) = 0$ to get roots $r = -2, -1, 1$. So, the general solution is $y(t) = C_1e^{-2t} + C_2e^{-t} + C_3e^{t}$.

(b) The characteristic equation here is $r^3 + 1 = 0$, or $(r+1)(r^2 - r + 1) = 0$ ($r = -1$ is fairly clearly a root, and at worst we can do polynomial long division, if the formula for a sum of cubes doesn't spring to mind). The roots of the quadratic term are $r = \frac{1}{2} \pm \frac{1}{2}i\sqrt{3}$, so our general solution is

$$y(t) = C_1e^{-t} + C_2e^{t/2}\cos(t\sqrt{3}/2) + C_3e^{t/2}\sin(t\sqrt{3}/2)$$

(c) The general solution here involves the root $r = -2$ with multiplicity one, the root $r = -4$ with multiplicity two, and $r = -1 \pm i$ with multiplicity two. Thus, the solution is

$$y(t) = C_1e^{-2t} + (C_2 + C_3t)e^{-4t} + e^{-t}\left(C_4\cos(t) + C_5\sin(t) + C_6t\cos(t) + C_7t\sin(t)\right).$$

(d) The characteristic equation here is $r^4 - 1 = (r+1)(r-1)(r^2+1) = 0$, so $r = \pm1, \pm i$ are our roots, all with multiplicity one. Thus, the homogeneous solution is $y_h(t) = C_1e^{t} + C_2e^{-t} + C_3\cos(t) + C_4\sin(t)$. For the particular solution, we use the method of undetermined coefficients, guessing $y_p(t) = (At^2 + Bt)e^{-t}$ (we don't need a e^{-t} term because it is in the homogeneous solution, but we do need a t^2 because $r = -1$ is a root). Then $y_p^{(4)}(t) = e^{-t}\left(At^2 - 8At + 12A + Bt - 4B\right)$, so in the DE we get

$$e^{-t}\left(12A - 4B - 8At\right).$$

So, $A = \frac{-1}{8}$ and $B = \frac{-3}{8}$, for a final solution of

$$y(t) = C_1 e^t + C_2 e^{-t} + C_3 \cos(t) + C_4 \sin(t) - \frac{1}{8}\left[t^2 e^{-t} + 3te^{-t}\right].$$