Spring Systems

The Punch Line: One of the most common DE models is the linear mass-spring system.

Setup: The mass-spring-damper system has the general form $mu'' + \gamma u' + ku = F(t)$, where m, γ, and k are the mass, damping, and spring constants, respectively, and $F(t)$ is an external force. We divide these models into three classes based on the discriminant $\gamma^2 - 4mk$: *underdamped* if $\gamma < \sqrt{4mk}$ (complex roots), *critically damped* if $\gamma = \sqrt{4mk}$ (repeated root), and *overdamped* if $\gamma > \sqrt{4mk}$ (distinct real roots).

1: (Note: in the given units, the gravitational acceleration is 9.8 meters per second squared.)

(a) A mass of 10 kilograms is placed on a scale supported by a spring, pushing it down by 1.4 meters (it is a very large scale). Material imperfections in the spring cause it to resist the motion of the mass with a force of 240 newtons (kilogram meters per second squared) when it has a velocity of 3 meters per second squared.

What is the general solution to this DE? Is it under-, over-, or critically damped?

(b) A particular spring has a spring constant of 100 newtons per meter (kilograms per second squared) and damping constant of 60 newtons per meter per second (kilograms per second). When a particular mass is suspended from this spring, it stretches by 98 centimeters.

What is the general solution to this DE? Is it under-, over-, or critically damped?

(c) A mass of 2 kilograms is suspended from a spring, stretching it by 2.45 meters. You are in charge of choosing a damping constant to make this system critically damped. What damping constant do you choose? What is the general solution to the resulting differential equation?

(a) We use the differential equation $mu'' + \gamma u' + ku = 0$. Here, $m = 10$, $3\gamma = 240$, and $10(9.8) = k(1.4)$, or $\gamma = 80$ and $k = 70$. So, the differential equation is $10u'' + 80u' + 70u = 0$, or

$$u'' + 8u' + 7u = 0.$$

The characteristic equation is $r^2 + 8r + 7 = 0$, or $(r+1)(r+7)$. This has two real roots, so we are overdamped (we can also examine the discriminant $(80)^2 - 4(10)(70) = 3600 > 0$). The general solution is $u(t) = C_1 e^{-t} + C_2 e^{-7t}$.

(b) Here we are given $k = 100$ and $c = 60$. We know $9.8m = 100/0.98$, or $m = 10$ kilograms. Then our differential equation is

$$10u'' + 60u' + 100u = 0.$$

Then we get $r^2 + 6r + 10 = 0$ as our characteristic equation, or $r = -3 \pm i$. So, the general solution is $u(t) = e^{-3t}(A\cos(t) + B\sin(t))$. This system is underdamped.

(c) We can solve for k first: $9.8(2) = 2.45k$, or $k = 8$. Then our differential equation is $2u'' + cu' + 8u = 0$. This is critically damped if $c = \sqrt{4(2)(8)} = \sqrt{64} = 8$. Then our DE is $2u'' + 8u' + 8u = 0$, or $u'' + 4u' + 4u = 0$. So, the characteristic equation is $-2 \pm \sqrt{4 - 16} = -2 \pm i\sqrt{12}$. The solution is then $u(t) = e^{-2t}\left(A\cos(t\sqrt{12}) + B\sin(t\sqrt{12})\right)$.