Systems of Equations

The Punch Line: The behavior of ODEs can be captured by systems of first-order equations.

Setup: Given a differential equation, we can introduce new variables for the various lower-order derivatives in order to write an equivalent system of equations, all of which are first-order. This system will have the form

$$x'(t) = P(t)x(t) + g(t)$$

(where x and g are vector-valued and P is matrix-valued).

1: Write the following DEs as an equivalent system of first-order equations (it would be instructive to write it in matrix form, if possible).

(a) $u'' + 5u' + 6u = 0$

(b) $2u'' + 12u' + 18u = t^2 e^{3t}$

(c) $t^3 y''' - 6ty' + 12y = 2$

(d) $\left[e^t y' + e^{-t} y\right]' = 0$

(e) $\left[t^2 u\right]'' = u$

(f) $u^2 + (u')^2 = u''$

Solving (Simple) Linear Systems: If we have a homogeneous linear system with constant coefficients $x' = Ax$, and A has n (the dimension of x) real distinct eigenvalues $\lambda_1, \dots, \lambda_n$ with eigenvectors $v_1, \dots, v_n$ respectively, the general solution to the DE is $x(t) = C_1 e^{\lambda_1 t} v_1 + \cdots + C_n e^{\lambda_n t} v_n$ where $C_1, \dots, C_n$ are constants (compare with a linear, constant coefficient n^{th} degree equation).

2: Find the general solutions to these equations:

(a) $x' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x$

(b) $x' = \begin{bmatrix} -2 & -1 \\ -2 & -3 \end{bmatrix} x$

(c) $x' = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 18 & 1 \end{bmatrix} x$

(d) $x' = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$