Math 4B
Worksheet 8

Systems of Equations

The Punch Line: The behavior of ODEs can be captured by systems of first-order equations.

Setup: Given a differential equation, we can introduce new variables for the various lower-order derivatives in order to write an equivalent system of equations, all of which are first-order. This system will have the form

$$x'(t) = P(t)x(t) + g(t)$$

(where x and g are vector-valued and P is matrix-valued).

1: Write the following DEs as an equivalent system of first-order equations (it would be instructive to write it in matrix form, if possible).

(a) $u'' + 5u' + 6u = 0$

(b) $2u'' + 12u' + 18u = t^2 e^{3t}$

(c) $t^3 y''' - 6ty' + 12y = 2$

(d) $\left[e^t y' + e^{-t} y\right]' = 0$

(e) $\left[t^2 u\right]'' = u$

(f) $u^2 + (u')^2 = u''$

(a) We put $v_1 = u$ and $v_2 = u'$, so we get the system of equations

$$\begin{cases} v_1' &= v_2, \\ v_2' &= -5v_2 - 6v_1. \end{cases}$$

As a matrix, this is

$$v' = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = P\vec{v}.$$

(b) Using the same substitutions as before, we get

$$v' = \begin{bmatrix} 0 & 1 \\ -9 & -6 \end{bmatrix} v + \begin{bmatrix} 0 \\ \frac{1}{2} t^2 e^{3t} \end{bmatrix}.$$

(c) Putting $x_1 = y$, $x_2 = y'$, and $x_3 = y''$, we get

$$x' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12t^{-3} & 6t^{-2} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 2t^{-3} \end{bmatrix}.$$

(d) We first evaluate the left hand side to get the DE $e^t y'' + (e^t + e^{-t})y' - e^{-t} y = 0$. From here, we get the matrix equation

$$x' = \begin{bmatrix} 0 & 1 \\ -e^{-2t} & -1 - e^{-2t} \end{bmatrix} x.$$

(e) Evaluating the DE gives $\left[2tu + t^2 u'\right]' = 2u + 4tu' + t^2 u'' = u$, or $t^2 u'' + 4tu' + u = 0$. This gives the matrix system

$$v' = \begin{bmatrix} 0 & 1 \\ -t^{-2} & -4t^{-1} \end{bmatrix} v.$$

(f) We can rewrite this as the system

$$\begin{cases} v_1' = v_2, \\ v_2' = v_1^2 + v_2^2. \end{cases}$$

However, as this is nonlinear we cannot write it as a matrix system.

Solving (Simple) Linear Systems: If we have a homogeneous linear system with constant coefficients $x' = Ax$, and A has n (the dimension of x) real distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ with eigenvectors $v_1, \ldots, v_n$ respectively, the general solution to the DE is $x(t) = C_1 e^{\lambda_1 t} v_1 + \cdots + C_n e^{\lambda_n t} v_n$ where $C_1, \ldots, C_n$ are constants (compare with a linear, constant coefficient n^{th} degree equation).

2: Find the general solutions to these equations:

(a) $x' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x$

(b) $x' = \begin{bmatrix} -2 & -1 \\ -2 & -3 \end{bmatrix} x$

(c) $x' = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 18 & 1 \end{bmatrix} x$

(d) $x' = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

(a) We start by computing the characteristic polynomial $\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = (\lambda + 1)(\lambda - 1)$. The eigenvalues are the roots of this, $\lambda = \pm 1$. For the positive eigenvalue $\lambda_1 = 1$, we get an eigenvector from the kernel of

$$A - \lambda_1 I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Such a vector is $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Similarly, for the negative eigenvalue $\lambda_2 = -1$, our eigenvector is from the kernel of $A - \lambda_2 I = A + I$, such as $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Then, our general solution is

$$x(t) = C_1 e^t v_1 + C_2 e^{-t} v_2 = \begin{bmatrix} C_1 e^t + C_2 e^{-t} \\ C_1 e^t - C_2 e^{-t} \end{bmatrix}.$$

(b) Here the characteristic polynomial is $(-2-\lambda)(-3-\lambda) - (-1)(-2) = \lambda^2 + 5\lambda + 4 = (\lambda+1)(\lambda+4)$, with roots $\lambda = -1, -4$. An eigenvector of $A + I$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and of $A + 4I$ is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. So, our general solution is

$$x(t) = C_1 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^{-4t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

(c) Here the characteristic polynomial is

$$(5-\lambda) \begin{vmatrix} 4-\lambda & 3 \\ 18 & 1-\lambda \end{vmatrix} = (5-\lambda)\big((4-\lambda)(1-\lambda) - (3)(18)\big) = (5-\lambda)(\lambda^2 - 5\lambda - 50) = (5-\lambda)(\lambda-10)(5+\lambda).$$

So, our eigenvalues are $\lambda = 10, \pm 5$. We can see that the vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an eigenvector for $\lambda = 5$. For $\lambda = -5$ we see that $\begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$ works, and for $\lambda = 10$ we see that $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ works. So, our general solution is

$$x(t) = C_1 e^{5t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + C_2 e^{10t} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + C_3 e^{-5t} \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} C_1 e^{5t} \\ C_2 e^{10t} + C_3 e^{-5t} \\ 2C_2 e^{10t} - 3C_3 e^{-5t} \end{bmatrix}.$$

(d) Here the characteristic polynomial is $(1-\lambda)^2 - 1 = \lambda^2 - 2\lambda$, so the eigenvalues are $\lambda = 0, 2$. We can see that $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is in the kernel (0-eigenspace), and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has eigenvalue 2, so our general solution is

$$x(t) = C_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} C_2 e^{2t} + C_1 \\ C_2 e^{2t} - C_1 \end{bmatrix}.$$