Complex Eigenvalues

The Punch Line: The real and imaginary parts of a complex solution to a DE with complex eigenvalues are two real-valued solutions.

Setup: Given the linear constant-(matrix-)coefficient homogeneous equation $x' = Ax$, if $\lambda = \alpha + i\beta$ is an eigenvalue of A with eigenvector v, then $\mathfrak{Re}\left[e^{\alpha t}\left(\cos(\beta t) + i\sin(\beta t)\right)v\right]$ and $\mathfrak{Im}\left[e^{\alpha t}\left(\cos(\beta t) + i\sin(\beta t)\right)v\right]$ (the real and imaginary parts) are real-valued solutions to the DE.

1: Solve the following DEs (if initial conditions are given, use them, otherwise give the general solution):

(a) $x' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

(b) $x' = \begin{bmatrix} -2 & 1 \\ -2 & 0 \end{bmatrix}$

(c) $x' = \begin{bmatrix} 0 & 8 \\ -2 & 0 \end{bmatrix}$ and $x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2: An object weighing 4 pounds stretches a spring by 8 inches to reach equilibrium. When the object has a velocity of 8 inches per second, the spring exerts a damping force of 1/2 pounds on it.

(Use 32 feet per second squared as the acceleration due to gravity.)

(a) Write a homogeneous, second order linear differential equation describing the motion of the spring system.

(b) Re-write the equation as a homogeneous first order matrix differential equation.

(c) Find the general solution of the matrix equation.

(d) Write the vector initial condition and find the particular solution for the situation where the object is given an initial velocity of 48 feet per second in the positive direction from its equilibrium position.