Complex Eigenvalues

The Punch Line: The real and imaginary parts of a complex solution to a DE with complex eigenvalues are two real-valued solutions.

Setup: Given the linear constant-(matrix-)coefficient homogeneous equation $x' = Ax$, if $\lambda = \alpha + i\beta$ is an eigenvalue of A with eigenvector v, then $\mathfrak{Re}\left[e^{\alpha t}\left(\cos(\beta t) + i\sin(\beta t)\right)v\right]$ and $\mathfrak{Im}\left[e^{\alpha t}\left(\cos(\beta t) + i\sin(\beta t)\right)v\right]$ (the real and imaginary parts) are real-valued solutions to the DE.

1: Solve the following DEs (if initial conditions are given, use them, otherwise give the general solution):

(a) $x' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (b) $x' = \begin{bmatrix} -2 & 1 \\ -2 & 0 \end{bmatrix}$ (c) $x' = \begin{bmatrix} 0 & 8 \\ -2 & 0 \end{bmatrix}$ and $x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(a) We begin by computing the characteristic equation

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0.$$

So, $\lambda = \pm i$ are our eigenvalues. Using the positive i root, we investigate the kernel of $\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}$. The second row is $-i$ times the first, so we only need to use the equation $-iv_1 + v_2 = 0$ from the first row to conclude that $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is an eigenvector. So, we get the eigenvector $v = \begin{bmatrix} 1 \\ i \end{bmatrix}$ for eigenvalue i.

We compute

$$(\cos(t) + i\sin(t))\,v = \begin{bmatrix} \cos(t) + i\sin(t) \\ i\cos(t) - \sin(t) \end{bmatrix} = \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} + i\begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}.$$

Taking the real and imaginary parts, we see that $x_1(t) = \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}$ and $x_2(t) = \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$ must be two real-valued solutions to the DE. So, the general solution is

$$x(t) = C_1\begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} + C_2\begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}.$$

(b) Here, the characteristic equation is

$$\begin{vmatrix} -2-\lambda & 1 \\ -2 & -\lambda \end{vmatrix} = (\lambda + 2)\lambda + 2 = \lambda^2 + 2\lambda + 2 = 0.$$

Applying the Quadratic Formula gives $\lambda = \frac{-2\pm\sqrt{4-4(2)}}{2} = -1 \pm i$. Seeking the eigenvectors of $\lambda = -1 + i$, we examine the kernel of $\begin{bmatrix} -1-i & 1 \\ -2 & 1-i \end{bmatrix}$. Multiplying the first row by $-1+i$ gives the matrix $\begin{bmatrix} 2 & -1+i \\ -2 & 1-i \end{bmatrix}$, and now the second row is clearly a multiple of the first. So, we only need the equation $2v_1 + (-1+i)v_2 = 0$, or $v_1 = \frac{1-i}{2}v_2$ to conclude v is an eigenvector. So $v = \begin{bmatrix} 1-i \\ 2 \end{bmatrix}$ is an eigenvector.

We compute

$$e^{-t}(\cos(t)+i\sin(t))v = e^{-t}\begin{bmatrix}\cos(t)+i\sin(t)-i\cos(t)+\sin(t)\\ 2\cos(t)+2i\sin(t)\end{bmatrix} = e^{-t}\left(\begin{bmatrix}\cos(t)+\sin(t)\\ 2\cos(t)\end{bmatrix}+i\begin{bmatrix}\sin(t)-\cos(t)\\ 2\sin(t)\end{bmatrix}\right).$$

So, the general solution is

$$x(t) = C_1 e^{-t}\begin{bmatrix}\cos(t)+\sin(t)\\ 2\cos(t)\end{bmatrix} + C_2 e^{-t}\begin{bmatrix}\sin(t)-\cos(t)\\ 2\sin(t)\end{bmatrix}.$$

(c) Here the characteristic equation is

$$\begin{vmatrix}-\lambda & 8\\ -2 & -\lambda\end{vmatrix} = \lambda^2+16 = 0.$$

So, our roots are $\lambda = \pm 4i$. Using the positive root, we seek an eigenvector in the kernel of $\begin{bmatrix}-4i & 8\\ -2 & -4i\end{bmatrix}$. Multiplying the top row by $i/4$ we get $\begin{bmatrix}1 & 2i\\ -2 & -4i\end{bmatrix}$. Now the bottom row is clearly a multiple of the top row, so we only need the condition that $v_1 + 2iv_2 = 0$, or $v_1 = -2iv_2$. So, $v = \begin{bmatrix}-2i\\ 1\end{bmatrix}$ is an eigenvector.

We compute

$$(\cos(4t)+i\sin(4t))v = \begin{bmatrix}-2i\cos(4t)+2\sin(4t)\\ \cos(4t)+i\sin(4t)\end{bmatrix} = \begin{bmatrix}2\sin(4t)\\ \cos(4t)\end{bmatrix} + i\begin{bmatrix}-2\cos(4t)\\ \sin(4t)\end{bmatrix}.$$

So, our general solution is

$$x(t) = C_1\begin{bmatrix}2\sin(4t)\\ \cos(4t)\end{bmatrix} + C_2\begin{bmatrix}-2\cos(4t)\\ \sin(4t)\end{bmatrix}.$$

Then $x(0) = \begin{bmatrix}-2C_2\\ C_1\end{bmatrix}$, so we see that we must have $C_2 = \frac{-1}{2}$ and $C_1 = 1$ to meet the initial condition. So, the solution to the IVP is

$$x(t) = \begin{bmatrix}2\sin(4t)\\ \cos(4t)\end{bmatrix} - \frac{1}{2}\begin{bmatrix}-2\cos(4t)\\ \sin(4t)\end{bmatrix} = \begin{bmatrix}\cos(4t)+2\sin(4t)\\ \cos(4t)-\frac{1}{2}\sin(4t)\end{bmatrix}.$$

2: An object weighing 4 pounds stretches a spring by 8 inches to reach equilibrium. When the object has a velocity of 8 inches per second, the spring exerts a damping force of 1/2 pounds on it.

(Use 32 feet per second squared as the acceleration due to gravity.)

(a) Write a homogeneous, second order linear differential equation describing the motion of the spring system.

(b) Re-write the equation as a homogeneous first order matrix differential equation.

(c) Find the general solution of the matrix equation.

(d) Write the vector initial condition and find the particular solution for the situation where the object is given an initial velocity of 48 feet per second in the positive direction from its equilibrium position.

(a) We first use the relations $m = W/g$, $k = W/\ell$, and $\gamma = F_{\text{damping}}/v$ to determine the relevant constants. The first gives $m = 1/8$ pound seconds squared per foot (yes, this is a unit of mass), the second gives $k = 1/2$ pounds per inch, or $k = 6$ pounds per foot, and the third gives $\gamma = 1/16$ pound seconds per inch, or 3/4 pound seconds per foot. So, our differential equation is $\frac{1}{8}u'' + \frac{3}{4}u' + 6u = 0$, which we can re-write to give a coefficient of one on the highest derivative:

$$u'' + 6u' + 48u = 0.$$

(b) Introducing the variable $v = u'$ and putting $x = \begin{bmatrix} u \\ v \end{bmatrix}$ allows us to write our initial equation as the system

$$\begin{cases} u' &= v, \\ v' &= -48u - 6v, \end{cases}$$

or the matrix equation

$$x' = \begin{bmatrix} 0 & 1 \\ -48 & -6 \end{bmatrix} x.$$

(c) Our first task is to find the eigenvalues of the matrix. We compute

$$\begin{vmatrix} -\lambda & 1 \\ -48 & -6-\lambda \end{vmatrix} = \lambda(\lambda+6) + 48 = \lambda^2 + 6\lambda + 48 = 0.$$

Then the Quadratic Formula gives us that $\lambda = \frac{-6 \pm \sqrt{36-192}}{2} = -3 \pm i\sqrt{39}$. Let's write $\omega = \sqrt{39}$ for neatness.

To find an eigenvector, we look at the kernel of $\begin{bmatrix} 3 - i\omega & 1 \\ -48 & -3-i\omega \end{bmatrix}$. Multiplying the first row by $3 + i\omega$ (and using $\omega^2 = 39$) gives $\begin{bmatrix} 48 & 3+i\omega \\ 48 & -3-i\omega \end{bmatrix}$. Clearly, the second row is a multiple of the first, so our condition to be an eigenvector is that $48v_1 + (3+i\omega)v_2 = 0$. So, $v = \begin{bmatrix} 3+i\omega \\ -48 \end{bmatrix}$ is an eigenvector.

We examine

$$\begin{aligned} e^{-3t}(\cos(\omega t) + i\sin(\omega t))v &= e^{-3t}\begin{bmatrix} 3\cos(\omega t) + 3i\sin(\omega t) + i\omega\cos(\omega t) - \omega\sin(\omega t) \\ -48\cos(\omega t) - 48i\sin(\omega t) \end{bmatrix} \\ &= e^{-3t}\begin{bmatrix} 3\cos(\omega t) - \omega\sin(\omega t) \\ -48\cos(\omega t) \end{bmatrix} + ie^{-3t}\begin{bmatrix} \omega\cos(\omega t) + 3\sin(\omega t) \\ -48\sin(\omega t) \end{bmatrix}. \end{aligned}$$

So, our general solution is

$$x(t) = C_1 e^{-3t}\begin{bmatrix} 3\cos(\omega t) - \omega\sin(\omega t) \\ -48\cos(\omega t) \end{bmatrix} + C_2 e^{-3t}\begin{bmatrix} 3\sin(\omega t) + \omega\cos(\omega t) \\ -48\sin(\omega t) \end{bmatrix}.$$

(d) Our vector initial condition is $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (it's important to note that we have written all constants with feet as our length units, so we don't have to convert units here). In the general solution, we have

$$x(0) = C_1 \begin{bmatrix} 3 \\ -48 \end{bmatrix} + C_2 \begin{bmatrix} \omega \\ 0 \end{bmatrix}.$$

So, we must satisfy the two equations

$$\begin{cases} 3C_1 + \omega C_2 & = 0 \\ -48C_1 = 48. \end{cases}$$

Clearly, we need $C_1 = -1$, so the first equation becomes $\omega C_2 = 3$, or $C_2 = 3\omega^{-1}$. Then our final solution is

$$x(t) = e^{-3t} \begin{bmatrix} \omega \sin(\omega t) - 3\cos(\omega t) \\ 48\cos(\omega t) \end{bmatrix} + e^{-3t} \begin{bmatrix} 3\cos(\omega t) + \frac{9}{\omega} \sin(\omega t) \\ -\frac{144}{\omega} \sin(\omega t) \end{bmatrix} = e^{-3t} \begin{bmatrix} \frac{48}{\omega} \sin(\omega t) \\ 48\cos(\omega t) - \frac{144}{\omega} \sin(\omega t) \end{bmatrix}.$$