Green's Theorem

The Punch Line: We get the same result by measuring local rotation in the interior of a region or global rotation around the boundary.

If $D \subset \mathbb{R}^2$ is a (nice) region bounded by the simple closed curve $\vec{c}\colon [a,b] \to \mathbb{R}^2$, oriented such that the region lies always on the left hand side of the curve, then

$$\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_{\vec{c}} P\,dx + Q\,dy = \iint_D \operatorname{curl}(\vec{F})\,dA$$

for all $\mathcal{C}^1$ vector fields $\vec{F} = (P, Q)$. Here $\operatorname{curl}(\vec{F}) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix}$ is the scalar curl of the vector field $\vec{F}$. It is a notion of derivative that measures the "local rotation" of the vector field at each point.

Computational

(a) Suppose $\vec{F} = x^2\hat{\imath} + xy\hat{\jmath}$. Compute $\operatorname{curl}(\vec{F})$, and verify Green's Theorem for $D = \{(x,y) : 0 \le x \le 1, 0 \le y \le 1\}$.

(b) Use Green's Theorem to compute $\int_{\vec{c}} y^3\,dx + x^3\,dy$ where $\vec{c}$ is the triangular path from the origin to $(R,0)$ to $(0,R)$ back to the origin (oriented positively).

(a) We have $P = x^2$ and $Q = xy$, so $\operatorname{curl}(\vec{F}) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y$. Then we can compute

$$\begin{aligned} \iint_D \operatorname{curl}(\vec{F})\,dA &= \int_0^1 \int_0^1 y\,dx\,dy \\ &= \int_0^1 y\,dy \\ &= \left[\frac{1}{2}y^2\right]_0^1 = \frac{1}{2}. \end{aligned}$$

We can parametrize the boundary as the straight line from $(0,0)$ to $(1,0)$, the line from $(1,0)$ to $(1,1)$, the line from $(1,1)$ to $(0,1)$, and the line from $(0,1)$ to $(0,0)$. That is, as $(t,0)$ followed by $(1,t)$ followed by $(1-t,1)$ followed by $(0,1-t)$, each of which with $0 \le t \le 1$. This then gives us that

$$\begin{aligned} \int_{\vec{c}} \vec{F}\,d\vec{s} &= \int_0^1 (t^2,0)\cdot(1,0)\,dt + \int_0^1 (1,t)\cdot(0,1)\,dt + \int_0^1 \left((1-t)^2, 1-t\right)\cdot(-1,0)\,dt + \int_0^1 (0,0)\cdot(0,-1)\,dt \\ &= \int_0^1 t^2 + t - (1-t)^2\,dt = \int_0^1 3t - 1\,dt = \left[\frac{3}{2}t^2 - t\right]_0^1 = \frac{1}{2}. \end{aligned}$$

(b) We see $\operatorname{curl}(\vec{F}) = 3(y^2 - x^2)$. Thus, we have by Green's Theorem

$$\begin{aligned} \int_{\vec{c}} \vec{F}\,d\vec{s} &= 3\int_0^R \int_0^{R-y} y^2 - x^2\,dx\,dy \\ &= \int_0^R 3y^2(R-y) - (R-y)^3\,dy = \int_0^R -2y^3 + 3R^2y - R^3\,dy \\ &= \left[-\frac{1}{2}y^4 + \frac{3}{2}R^2y^2 - R^3y\right]_0^R = 0. \end{aligned}$$

We can actually make a nice symmetry argument here: exchanging x and y in the integrand picks up a negative sign. The triangle we are integrating over is symmetric with respect to exchanging x and y, however, so the integral has to vanish, as this operation doesn't change the region (so the integral should have the same value), but negates the integrand (so the integral should have the negative of its original value), and 0 is the only number equal to its own negative.

In fact, we could make another symmetry argument on the path integral. The integrals along the legs of the triangle must vanish, as $P(x,0) = Q(0,y) = 0$. Since $P(x,y) = Q(y,x)$, exchanging x and y in the integrands should not change the value of a line integral. However, exchanging x and y along the hypoteneuse integrates along the same line in the opposite direction, picking up a negative sign. Again, we need a number equal to its own negative.

Theoretical

(a) What is $\int_{\vec{C}} P(x)\,dx + Q(y)\,dy$?

(b) Suppose we wanted to compute $\iint_D (x^2+y^2)\,dA$ (integrals of this form are related to the *moment of inertia* in physics) for some region D. Can we find a vector field $\vec{F}$ such that we can compute this by taking an integral around the boundary of D?

(a) Since $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} = 0$, we have $\operatorname{curl}(\vec{F}) = 0$, so the loop integral is also zero by Green's Theorem.

(b) We need a vector field $\vec{F} = P\hat{\imath} + Q\hat{\jmath}$ such that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = x^2 + y^2$. One way to do this is to say $\frac{\partial Q}{\partial x} = x^2$ and $-\frac{\partial P}{\partial y} = y^2$, and supposing $P = P(y)$ and $Q = Q(x)$, so these are actually full derivatives. Then we can solve the ODEs to get $P(y) = -\frac{1}{3}y^3$ and $Q(x) = \frac{1}{3}x^3$, so $\vec{F} = -\frac{1}{3}\left(y^3\hat{\imath} - x^3\hat{\jmath}\right)$.

Alternately, we could imagine that $\frac{\partial Q}{\partial x} = y^2$ and $-\frac{\partial P}{\partial y} = x^2$, and allow them to depend on both variables. Then one solution to the PDEs is $P(x,y) = -x^2y$ and $Q(x,y) = xy^2$, or $\vec{F} = -xy\,(x\hat{\imath} - y\hat{\jmath})$.