Green's Theorem

The Punch Line: We get the same result by measuring local rotation in the interior of a region or global rotation around the boundary.

If $D \subset \mathbb{R}^2$ is a (nice) region bounded by the simple closed curve $\vec{c}: [a,b] \to \mathbb{R}^2$, oriented such that the region lies always on the left hand side of the curve, then

$$\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_{\vec{c}} P\,dx + Q\,dy = \iint_D \operatorname{curl}(\vec{F})\,dA$$

for all $\mathcal{C}^1$ vector fields $\vec{F} = (P, Q)$. Here $\operatorname{curl}(\vec{F}) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix}$ is the scalar curl of the vector field $\vec{F}$. It is a notion of derivative that measures the "local rotation" of the vector field at each point.

Computational

(a) Suppose $\vec{F} = x^2\hat{\imath} + xy\hat{\jmath}$. Compute $\operatorname{curl}(\vec{F})$, and verify Green's Theorem for $D = \{(x,y) : 0 \le x \le 1, 0 \le y \le 1\}$.

(b) Use Green's Theorem to compute $\int_{\vec{c}} y^3\,dx + x^3\,dy$ where $\vec{c}$ is the triangular path from the origin to $(R,0)$ to $(0,R)$ back to the origin (oriented positively).

Theoretical

(a) What is $\int_{\vec{c}} P(x)\,dx + Q(y)\,dy$?

(b) Suppose we wanted to compute $\iint_D (x^2+y^2)\,dA$ (integrals of this form are related to the *moment of inertia* in physics) for some region D. Can we find a vector field $\vec{F}$ such that we can compute this by taking an integral around the boundary of D?