Divergence and Stokes' Theorem

The Punch Line: We can also compute flux and circulation by looking at boundary values in three dimensions.

For a region $W \subset \mathbb{R}^3$ with boundary surface S, and vector field $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$, we have $\iint_S \vec{F} \cdot \hat{n}\, dA = \iiint_W \text{div}(\vec{F})\, dV$, where $\hat{n}$ is the outward unit normal vector, and $\text{div}(\vec{F}) = \nabla \cdot \vec{F}$ is the divergence.

For a surface $S \subset \mathbb{R}^3$ with positive boundary curve $\vec{c}$, we have $\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dA$, where the orientation of $\hat{n}$ depends on the orientation of $\vec{c}$.

Computational Let W be the region in $\mathbb{R}^3$ given in polar coordinates by

$$W = \left\{(r, \theta, z) : 0 \le r \le z^4,\ 0 \le z \le 1,\ 0 \le \theta \le 2\pi\right\},$$

with boundary $S = S_1 \cup S_2$ for S_1 the lower "bowl" surface and S_2 the upper "flat disc" surface, with outward facing normals. Define $\vec{F} = (x - yz, xz - y, z - xy)$.

(a) Compute $\iint_{S_1} \vec{F} \cdot \hat{n}\, dA$.

(b) Compute $\iint_{S_1} (\nabla \times \vec{F}) \cdot \hat{n}\, dA$.

(c) Compute $\iint_{S_2} (\nabla \times \vec{F}) \cdot \hat{n}\, dA$.

(a) We first use the Divergence Theorem to write

$$\iint_{S_1} \vec{F} \cdot \hat{n}\, dA = \iiint_W \text{div}(\vec{F})\, dV - \iint_{S_2} \vec{F} \cdot \hat{n}\, dA.$$

To make use of this result, we compute $\text{div}(\vec{F}) = 1$, and note that on S_2, we have $\hat{n} = \hat{k}$, the unit vector parallel to the positive z axis. Then we compute

$$\iiint_W \text{div}(\vec{F})\, dV = \int_0^{2\pi} \int_0^1 \int_0^{z^4} r\, dr\, dz\, d\theta = \int_0^{2\pi} \int_0^1 \frac{1}{2} z^8\, dz\, d\theta = \int_0^{2\pi} \frac{1}{18}\, d\theta = \frac{\pi}{9},$$

$$\iint_{S_2} \vec{F} \cdot \hat{n}\, dA = \int_0^{2\pi} \int_0^1 (r\cos\theta - r\sin\theta, r\cos\theta - r\sin\theta, 1 - r^2\cos\theta\sin\theta) \cdot (0,0,1) r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 r - r^3 \cos\theta \sin\theta\, dr\, d\theta = \int_0^{2\pi} \frac{1}{2} - \frac{1}{8} \sin 2\theta\, d\theta = \pi.$$

Thus, the integral we are interested in is $\iint_{S_1} \vec{F} \cdot \hat{n}\, dA = \frac{\pi}{9} - \pi = -\frac{8}{9}\pi$.

(b) The curve $\vec{c}(\theta) = (\cos\theta, -\sin\theta, 1)$ for $0 \le \theta \le 2\pi$ traverses a circular path at height one in the clockwise direction, so it is a positively oriented boundary curve for S_1. It has tangent vector $\vec{c}'(\theta) = (-\sin\theta, -\cos\theta, 0)$. This allows us to use Stokes' Theorem to compute

$$\iint_{S_1} (\nabla \times \vec{F}) \cdot \hat{n}\, dA = \int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} (\cos\theta + \sin\theta, \cos\theta + \sin\theta, 1 + \cos\theta\sin\theta) \cdot (-\sin\theta, -\cos\theta, 0) d\theta$$

$$= \int_0^{2\pi} -\sin^2\theta - \cos^2\theta - 2\cos\theta\sin\theta\, d\theta = \int_0^{2\pi} -1 - \sin 2\theta\, d\theta = -2\pi.$$

(c) The curve $\vec{c}$ above has a negative orientation for the region S_2, so our answer is simply $\iint_{S_2} (\nabla \times \vec{F})\, dA = 2\pi$.

Theoretical

(a) Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function and $S(k) = \{\vec{x} : f(\vec{x}) = k\}$ denotes the level surface for value $k \in \mathbb{R}$. What is the unit normal vector to the surface $S(k)$? (There are two possible answers.)

(b) With the setup above, suppose $W(k) = \{\vec{x} : f(\vec{x}) \le k\}$ is a closed and bounded region (which then has boundary $S(k)$; this can arise with functions like $f(\vec{x}) = \left\|\vec{x}\right\|^2$). Compute $\iiint_{W(k)} \Delta f \, dV$ (where $\Delta f = \nabla \cdot \nabla f$) using the Divergence Theorem to conclude that it is nonnegative, and positive if k is not the maximum value of f.

(c) Use the Divergence Theorem to prove the following integration by parts formula for a closed and bounded region W with boundary S, and $f, g : \mathbb{R}^3 \to \mathbb{R}$ differentiable functions:

$$\iiint_W f \Delta g \, dV = \iint_S f \nabla g \cdot \hat{n} \, dA - \iiint_W \nabla f \cdot \nabla g \, dV.$$

You may need the product rule $\operatorname{div}(f\vec{F}) = \nabla f \cdot \vec{F} + f \operatorname{div}(\vec{F})$ for a differentiable scalar function f and vector field $\vec{F}$.

(a) We know that ∇f will be everywhere orthogonal to $S(k)$ for fixed k, because it is the direction along which the function changes. Thus, the unit normal vectors are $\pm \|\nabla f\|^{-1} \nabla f$. This of course has singularities if the gradient vanishes, requiring more involved analysis (beyond the intended scope of this exercise).

(b) Taking $\vec{F} = \nabla f$, we use the Divergence Theorem to compute (with the outward facing normal)

$$\iiint_{W(k)} \Delta f \, dV = \iint_{S(k)} \nabla f \cdot (\|\nabla f\|^{-1} \nabla f) \, dA = \iint_{S(k)} \|\nabla f\| \, dA$$

(assuming that on the surface any singularities are removable, and generally the integral makes sense; this can fail to be true if e.g. $f = k$ on some solid region with positive volume). The integrand on the right hand side is nonnegative, so the triple integral must be as well. If k is not the maximum value of f, for some $\vec{x} \in S(k)$ we will have $\nabla f(\vec{x}) \neq \vec{0}$, hence $\left\|\nabla f(\vec{x})\right\| > 0$, hence the integral will be positive. It's worth noting that if k is the maximum value of f, the region $S(k)$ will quite possibly not be a surface; we actually require stronger conditions on f than provided here for everything to be properly well-behaved, and it's worth thinking about what they are.

(c) We put $\vec{F} = f \nabla g$. Then $\operatorname{div}(\vec{F}) = \nabla f \cdot \nabla g + f \Delta g$ by the product rule above. This allows us to compute

$$\iint_S f \nabla g \cdot \hat{n} \, dA = \iiint_W \nabla f \cdot \nabla g + f \Delta g \, dV,$$

and rearranging gives the integration by parts formula.